Math 558 – Homework #4

March 23, 2007

1. Define $\aleph_0 = \mathrm{card}(\mathbb{N})$. Without using the Cantor/Schröder/Bernstein Theorem, prove that
$$\aleph_0 = \mathrm{card}(\mathbb{Z}) = \mathrm{card}(\mathbb{Q})$$
and
$$2^{\aleph_0} = \mathrm{card}(\mathbb{R}) = \mathrm{card}(\mathbb{C}).$$

2. Using Cantor/Schröder/Bernstein, prove that
$$\mathrm{card}(C[0,1]) = \mathrm{card}([0,1]) = \mathrm{card}([0,1] \times [0,1]) = 2^{\aleph_0}.$$
Here $[0,1] = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ is the closed unit interval, and $C[0,1]$ is the set of continuous real-valued functions on $[0,1]$.

3. Prove König's Theorem: If $\kappa_i < \lambda_i$ for all $i \in I$, then
$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i \,.$$

4. Give counterexamples showing the failure of
$$\sum_{i \in I} \kappa_i < \sum_{i \in I} \lambda_i$$
and
$$\prod_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i \,.$$

5. Prove the following laws of ordinal arithmetic.
   (a) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
   (b) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
   (c) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
   (d) $\alpha + 0 = 0 + \alpha = \alpha$, $\alpha \cdot 0 = 0 \cdot \alpha = 0$, $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$.

6. Give a counterexample showing the failure of
$$(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma \,.$$

7. Let α be an ordinal number, and let X be a nonempty set of ordinal numbers. Prove that
$$\alpha + \sup X = \sup\{\alpha + \beta \mid \beta \in X\}$$
and
$$\alpha \cdot \sup X = \sup\{\alpha \cdot \beta \mid \beta \in X\} \,.$$

8. Give counterexamples showing the failure of
$$(\sup X) + \beta = \sup\{\alpha + \beta \mid \alpha \in X\}$$
and
$$(\sup X) \cdot \beta = \sup\{\alpha \cdot \beta \mid \alpha \in X\} \,.$$