Math 558 – Homework #3

March 2, 2007

1. Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$ = the natural numbers,

   $$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \text{the integers},$$

   and $\mathbb{R} = (-\infty, \infty)$ = the real numbers.

   According to Matiyasevich's Theorem, we can find a polynomial

   $$f(w, x_1, \ldots, x_k)$$

   with integer coefficients, such that the set of $a \in \mathbb{N}$ for which the equation $f(a, x_1, \ldots, x_k) = 0$ has a solution in $\mathbb{N}$ is noncomputable.

   (a) Discuss the analogous question in which "solution in $\mathbb{N}$" is replaced by "solution in $\mathbb{Z}$".

   Hint: You may use the following well-known theorem of Lagrange: For each $n \in \mathbb{N}$ there exist $a, b, c, d \in \mathbb{N}$ such that

   $$n = a^2 + b^2 + c^2 + d^2 \, .$$

   (b) Discuss analogous questions in which "solution in $\mathbb{N}$" is replaced by "solution in $\mathbb{R}$".

2. Explain in detail how you would translate the following statements of Euclidean plane geometry into sentences of the language

   $$+, -, \cdot, 0, 1, <, =$$

   over the real number system.

   (a) For every two points, there is a unique line passing through them.

   (b) For every three non-collinear points, there is a unique circle passing through them.

   (c) For every line L and circle C, the intersection of L and C consists of at most two points.

   (d) For every circle C and point P lying on C, there exists one and only one line L such that $L \cap C = P$. (I.e., a tangent line.)

   (e) Every line segment has a unique midpoint.

   (f) Every angle can be uniquely bisected.

   (g) The three angle bisectors of any triangle meet in a single point.

   (h) Every angle can be uniquely trisected.