Math 558 – Homework #2

February 21, 2007

1. Find a primitive recursive function $f(x,y)$ such that, for all x and y,

$$\varphi^{(1)}_{f(x,y)} = \varphi^{(1)}_x \circ \varphi^{(1)}_y \, .$$

   Hint: Use the Parametrization Theorem.

2. A function $f : \mathbb{N}^k \to \mathbb{N}$ is said to be *limit recursive* if there exists a recursive function $g : \mathbb{N}^{k+1} \to \mathbb{N}$ such that, for all $x_1, \ldots, x_k$, $f(x_1, \ldots, x_k) = \lim_s g(x_1, \ldots, x_k, s)$.

   Prove that f is Δ^0_2 if and only if f is limit recursive.

3. Two sets $A, B \subseteq \mathbb{N}$ are said to be *recursively inseparable* if there is no recursive set $X \subseteq \mathbb{N}$ such that $A \subseteq X$ and $B \cap X = \emptyset$. Find a disjoint pair of Σ^0_1 sets which are recursively inseparable.

   Hint: Let $A = K_0$ and $B = K_1$, where $K_n = \{x \mid \varphi^{(1)}_x(x) \simeq n\}$.

4. Show that the following number-theoretic predicates are arithmetically definable, by exhibiting formulas which define them over the structure $(\mathbb{N}, +, \cdot, 0, 1, =)$.

   (a) $\mathrm{GCD}(x,y) = z$.

   (b) $\mathrm{LCM}(x,y) = z$.

   (c) $\mathrm{Quotient}(x,y) = z$.

   (d) $\mathrm{Remainder}(x,y) = z$.

   (e) $\beta(x,y,z) = w$.

   (f) x is the largest prime number less than y.

   (g) x is the product of all the prime numbers less than y.

5. Which of the following number-theoretic predicates are arithmetically definable? Prove your answers.

(a) x is the sum of all the prime numbers less than y.

(b) $x^y = z$.

(c) $x! = y$.

6. Find a pair of numbers r, a such that $\beta(r, a, 0) = 11$, $\beta(r, a, 1) = 19$, $\beta(r, a, 2) = 30$, $\beta(r, a, 3) = 37$, $\beta(r, a, 4) = 51$.

   (Hint: First find an appropriate a by hand. Then write a small computer program to find r by brute force.)

7. Let A and B be subsets of $\mathbb{N}$. Prove that if A is reducible to B and B is arithmetically definable, then A is arithmetically definable.

8. For each $n \geq 1$ let C_n be a subset of $\mathbb{N}$ which is Σ_n^0 complete. Consider the set $B = \{2^n 3^x \mid x \in C_n\}$. Prove that B is not arithmetically definable.

9. Prove that the set Fml of all Gödel numbers of formulas is primitive recursive.

10. Prove that the set Snt of all Gödel numbers of sentences is primitive recursive.