

Math 558 – Final Exam (= Midterm Exam #4)

May 7, 2007

(6 problems)

1. Outline a proof that if an inaccessible cardinal exists, then a countable transitive model of ZFC exists.
2. True or false.
 - (a) The GCH is a logical consequence of the axioms of ZFC.
 - (b) The negation of the GCH is consistent with ZFC.
 - (c) An ordinal α is an initial ordinal if and only if $\alpha = |\alpha|$.
 - (d) Given an infinite set X , the cardinality of the transitive closure of X is equal to the cardinality of X .
 - (e) If X is a set of ordinals, then $\bigcup X$ is an ordinal.
 - (f) For all limit ordinals α and β , $\alpha < \beta$ if and only if $\alpha + \gamma = \beta$ for some limit ordinal γ .
 - (g) For all infinite cardinals κ and λ , $\kappa < \lambda$ if and only if $\kappa + \mu = \lambda$ for some infinite cardinal μ .
 - (h) Viewing the ordered pair $(3, 15)$ as a pure well founded set, the rank of $(3, 15)$ is 17.
 - (i) If λ is a weakly inaccessible cardinal, then $\lambda = \aleph_\lambda$.
 - (j) For all infinite cardinals κ and λ , $\kappa + \lambda = \kappa \cdot \lambda$.
3. Let M be a countable transitive model of ZFC. Let P be a poset belonging to M , and let G be an M -generic filter on P . Thus $M[G]$ is again a countable transitive model of ZFC.
 - (a) A poset P is said to satisfy the λ -chain condition if every antichain in P is of cardinality $< \lambda$.

Let λ be an uncountable cardinal of M , and assume that $M \models$ “ P satisfies the λ -chain condition.”

Prove that the cardinals $\geq \lambda$ in $M[G]$ are the same as the cardinals $\geq \lambda$ in M .
 - (b) A poset P is said to be *countably closed* if every every descending ω -sequence in P has a lower bound in P .

Assume that $M \models$ “ P is countably closed.”

Prove that $\mathcal{P}(\omega) \cap M[G] = \mathcal{P}(\omega) \cap M$.
 - (c) Let P be the poset in M consisting of all functions $f \in M$ such that $M \models$ “ $\text{dom}(f)$ is a countable ordinal and $\text{rng}(f) \subseteq \mathcal{P}(\omega)$ ”, ordered by reverse inclusion.

Use (a) and (b) to prove that $M[G] \models \text{CH}$.

4. Let λ be an inaccessible cardinal. Prove that there exist arbitrarily large cardinals $\kappa < \lambda$ such that $(R_\kappa, \in|R_\kappa)$ is an elementary submodel of $(R_\lambda, \in|R_\lambda)$.
5. Assume the GCH.
 - (a) Prove that for all ordinals α , the cardinality of $R_{\omega+\alpha}$ is \aleph_α .
 - (b) Prove that for all infinite cardinals κ and λ ,

$$\lambda^\kappa = \begin{cases} \lambda & \text{if } \kappa < \text{cf}(\lambda), \\ \lambda^+ & \text{if } \text{cf}(\lambda) \leq \kappa \leq \lambda, \\ \kappa^+ & \text{if } \kappa \geq \lambda. \end{cases}$$

6. Recall that ω is the first infinite ordinal, and \mathbb{N} is the set of natural numbers. We have seen that $\omega = \mathbb{N}$.

Let X be a subset of ω . Recall from the first part of the course that X is *arithmetical* if and only if X is definable over $(\mathbb{N}, +, \cdot)$, i.e., $X \in \Sigma_n^0$ for some $n \in \omega$.

Outline a proof that X is arithmetical if and only if $X \in \text{Def}(R_\omega)$.