Math 558 – Spring 2004

Final Exam

May 3, 2004

1. Find a quantifier free formula which is equivalent over the real number system to the formula $\exists x\,(ax^3 + bx^2 + cx + d > 0)$.

2. Recall that Gödel's beta function is $\beta(r, m, i)$ = the remainder of r on division by $m \cdot (i + 1) + 1$.

   (a) State the crucial property of the beta function, which has been used in proving that every primitive recursive function is definable over $(\mathbb{N}, +, \cdot)$.

   (b) Find positive integers r and m such that

   $$\beta(r, m, 0) = 0, \;\; \beta(r, m, 1) = 1, \;\; \beta(r, m, 2) = 2.$$

3. Given a nonempty recursively enumerable set $A \subseteq \mathbb{N}$, show how to find a primitive recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $A = \mathrm{rng}(f)$.

4. Prove that there exists a primitive recursive function $f(m, n)$ such that

   $$\mathrm{dom}(\varphi^{(1)}_{f(m,n)}) \;=\; \mathrm{dom}(\varphi^{(1)}_m) \cup \mathrm{dom}(\varphi^{(1)}_n)$$

   for all $m, n \in \mathbb{N}$.

   Hint: You may use the Enumeration Theorem and the Parametrization Theorem.

5. True or False.

   (a) Given positive integers e and n, the problem of deciding whether the register machine program with Gödel number e halts in $\leq n$ steps is unsolvable.

   (b) Given a total recursive function $f : \mathbb{N} \to \mathbb{N}$, we can find a primitive recursive function $g : \mathbb{N} \to \mathbb{N}$ such that $f(n) < g(n)$ for all n.

   (c) If $A, B \subseteq \mathbb{N}$ are Π^0_2, then $A \cup B$ is Π^0_2.

   (d) There exists a recursively enumerable set $A \subseteq \mathbb{N}$ such that

   $$\{n \mid \exists i\, \forall j\; \left(2^i 3^j 5^n \in A\right)\}$$

   is Σ^0_3 complete.

(e) The cardinal $\aleph_{\omega+1}$ is uncountable and regular.

(f) If X and Y are infinite sets, then the cardinality of $X \cup Y$ is the same as the cardinality of $X \times Y$.

(g) R_ω satisfies all the axioms of ZFC except one.

(h) It is consistent with ZFC that, for all $n < \omega$, the cardinality of $R_{\omega+n}$ is $\aleph_n$.

(i) If there exists a transitive model of ZFC, then there exists a countable transitive model of $\mathsf{ZFC} + \forall\alpha\, (2^{\aleph_\alpha} = \aleph_{\alpha+1})$.

(j) If X is an infinite, pure, well founded set, then the transitive closure of X is a set of the same cardinality as X.

6. Let κ be an inaccessible cardinal. Show that there exists a limit ordinal δ such that $\omega < \delta < \kappa$ and R_δ is an elementary submodel of R_κ.

7. Let α be an ordinal number $> \omega$. Recall that R_α is the set of pure, well founded sets of rank $< \alpha$. Discuss the question of which axioms of ZFC are satisfied by R_α. How does this depend on α?