Math 558 – Spring 2004

Midterm Exam #2

April 9, 2004

1. Find positive integers a, r such that $\beta(a,r,0) = 0$, $\beta(a,r,1) = 1$, and $\beta(a,r,2) = 2$.

2. Find a quantifier free formula which is equivalent over the real number system to the formula $\exists x\,(x > 0 \wedge ax^3 + bx^2 + cx + d > 0)$.

3. True or False.

   (a) Every Σ^0_3 predicate on $\mathbb{N}$ is definable over the structure $(\mathbb{N}, +, \cdot)$.

   (b) There exists a Δ^0_3 subset of $\mathbb{N}$ which is neither Σ^0_2 nor Π^0_2.

   (c) Let $\mathbb{R}$ be the set of real numbers. If X is a subset of $\mathbb{R}$ which is definable over $(\mathbb{R}, +, \cdot)$, then X is the union of a finite set of points with a finite set of open intervals.

   (d) The set of integers, $\mathbb{Z}$, is definable over the structure $(\mathbb{R}, +, \cdot)$.

   (e) If κ is an initial von Neumann ordinal, then $\kappa = \sup\{|\alpha| \mid \alpha \in \kappa\}$.

   (f) If the GCH holds, then $\kappa^\kappa = \kappa^+$ for all infinite cardinals κ.

   (g) The set of irrational numbers is of cardinality $2^{\aleph_0}$.

   (h) The number of countable sets in the Euclidean plane is at most $\aleph_2$.

   (i) For every von Neumann ordinal α, we have $|\alpha| \in \alpha + 1$.

   (j) Every element of a von Neumann ordinal is a von Neumann ordinal.

4. Let κ be an infinite cardinal. The *cofinality* of κ is defined to be the smallest cardinal λ such that κ can be written as the sum of λ many cardinals, each of which is $< \kappa$. We write $\mathrm{cf}(\kappa) =$ the cofinality of κ. Prove that $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.

5. Let A be an infinite set of cardinality $\aleph_\alpha$. Let $W(A)$ be the set of all $R \subseteq A \times A$ such that (A, R) is a well ordering. Let $\equiv$ be the equivalence relation on $W(A)$ defined by putting $R_1 \equiv R_2$ if and only if (A, R_1) and (A, R_2) are of the same order type.

   (a) What is the cardinality of $W(A)$?

   (b) What is the cardinality of the set of equivalence classes, $W(A)/\!\equiv$?

   Justify your answers.