Math 558 – Spring 2004

Midterm Exam #1

February 18, 2004

1. Write a register machine program which computes the 1-place function $f(x) = \text{Rem}(x, 3) =$ the remainder of x on division by 3.

2. Prove that the set

$$T = \{e \mid \forall x\, \varphi_e^{(1)}(x)\downarrow\} = \{\text{indices of total recursive functions}\}$$

   is a complete Π_2^0 set.

3. True or False.

   (a) The complement of a Σ_2^0 set is necessarily Σ_2^0.

   (b) The complement of a Σ_2^0 set is necessarily Π_2^0.

   (c) The union or intersection of any two recursively enumerable sets is recursively enumerable.

   (d) There exists a 1-place total function which is computable but not primitive recursive.

   (e) If $f : \mathbb{N} \to \mathbb{N}$ is one-to-one, onto, and recursive, then the inverse function $f^{-1} : \mathbb{N} \to \mathbb{N}$ is recursive.

   (f) If $f : \mathbb{N} \to \mathbb{N}$ is one-to-one, onto, and primitive recursive, then the inverse function $f^{-1} : \mathbb{N} \to \mathbb{N}$ is primitive recursive.

   (g) Every subset of a recursively enumerable set is recursively enumerable.

   (h) Let $f : \mathbb{N} \to \mathbb{N}$ be a 1-place total recursive function. Let $A \subseteq \mathbb{N}$ be a recursively enumerable set. Let ψ be the restriction of f to A, i.e., ψ is the 1-place partial function given by

$$\psi(x) \simeq \begin{cases} f(x) & \text{if } x \in A, \\ \uparrow & \text{otherwise.} \end{cases}$$

   Then ψ is partial recursive.

   (i) Every nonempty recursively enumerable set is the range of a total recursive function.

(j) Every nonempty recursively enumerable set is the range of a primitive recursive function.

(k) Every infinite recursively enumerable set is the range of a one-to-one total recursive function.

(l) Every infinite recursively enumerable set is the range of a one-to-one primitive recursive function.

(m) A k-place partial function $\psi(x_1, \ldots, x_k)$ is partial recursive if and only if its graph $G_\psi = \{\langle x_1, \ldots, x_k, y\rangle \mid \psi(x_1, \ldots, x_k) \simeq y\}$ is Σ^0_1.

4. A real number x is said to be *computable* if the function $\lambda n\,(\,n\text{th digit of } x\,)$ is computable. For example, we have seen that $\sqrt{2}$ is a computable real number.

   Give a specific example of a real number which is not computable.

5. Find a pair of recursively enumerable sets $A, B \subseteq \mathbb{N}$ with the following properties:

   (a) $A \cap B = \emptyset$.

   (b) There is no recursive set $C \subseteq \mathbb{N}$ such that $A \subseteq C$ and $C \cap B = \emptyset$.

   Note: A set $C \subseteq \mathbb{N}$ is said to be recursive if and only if its characteristic function $\chi_C : \mathbb{N} \to \{0, 1\}$ is recursive.

   Hint: Consider the sets $A_i = \{e \mid \varphi_e^{(1)}(e) \simeq i\}$, where $i = 0, 1, 2, \ldots$.