Math 558 – Homework #1

January 30, 2008

1. Write a register machine program which computes the *exponential function*, i.e., the 2-place number-theoretic function $\exp(x, y) = x^y$. Note that $x^0 = 1$ for all x including $x = 0$.

2. Prove that the following 2-place number-theoretic functions and predicates are computable.

   (a) $\mathrm{Rem}(y, x) =$ the remainder of y upon division by x

   (b) $\mathrm{Quot}(y, x) =$ the quotient of y upon division by x

   (c) $\mathrm{GCD}(x, y) =$ the greatest common divisor of x and y

   (d) $\mathrm{LCM}(x, y) =$ the least common multiple of x and y

   (e) x is a divisor y

   (f) x is the largest prime divisor of y

3. Consider successive rational approximations of $\sqrt{2}$ given by Newton's method:

$$x_0 = 1, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   where $f(x) = x^2 - 2$. The first few values are $x_0 = 1$, $x_1 = 3/2$, $x_2 = 17/12$, $x_3 = 577/408$. Let $a(n)$ and $b(n)$ respectively be the numerator and denominator of x_n. Thus $a(n)$ and $b(n)$ are 1-place number-theoretic functions. The first few values of these functions are $a_0 = b_0 = 1$, $a_1 = 3$, $b_1 = 2$, $a_2 = 17$, $b_2 = 12$, $a_3 = 577$, $b_3 = 408$.

   Use primitive recursion to prove that the functions $a(n)$ and $b(n)$ are computable.

   Hint: Use the same method which was used to prove that the Fibonacci sequence is computable.

4. Prove that the 1-place number-theoretic function

$$f(n) = \text{the } n\text{th decimal digit of } \pi = 3.141592\cdots$$

is computable.

5. If f is a k-place number-theoretic function, define the *graph* of f to be the $k+1$-place number-theoretic predicate

$$P(x_1,\ldots,x_k,y) \qquad \equiv \qquad f(x_1,\ldots,x_k)=y.$$

   Prove that f is a computable function if and only if the graph of f is a computable predicate.

   To what extent does this generalize to partial functions?

6. If f is a k-place number-theoretic function, prove that f is computable if and only if the 1-place number-theoretic function

$$f^*(w) \;=\; f((w)_1,\ldots,(w)_k)$$

   is computable.

   To what extent does this generalize to partial functions?

7. Let $f:\mathbb{N}\overset{1-1\text{ onto}}{\longrightarrow}\mathbb{N}$ be a permutation of $\mathbb{N}$, the set of natural numbers. Prove that f is computable if and only if the inverse permutation f^{-1} is computable.

   To what extent does this generalize to partial functions?