

Math 557 – Midterm Exam #2

November 4, 2005

SOLUTIONS

1. Let $M = (U_M, f_M, g_M, I_M)$ where $U_M = \{0, 1, 2, 3, 4\}$, I_M is the identity relation on U_M , and f_M, g_M are the binary operations of addition and multiplication modulo 5. Thus M is essentially just the ring of integers modulo 5. Let L be the language consisting of f, g, I . Note that M is a normal L -structure.

Write an L -sentence A such that for all normal L -structures M' , M' satisfies A if and only if M' is isomorphic to M .

Solution. A brute force solution is to let A be $\exists x_0 \exists x_1 \exists x_2 \exists x_3 \exists x_4 B$, where B is the conjunction of $\{\forall y (Ix_0y \vee Ix_1y \vee Ix_2y \vee Ix_3y \vee Ix_4y)\} \cup \{\neg Ix_i x_j : i \neq j\} \cup \{Ifx_i x_j x_k \mid i + j = k \pmod{5}\} \cup \{Igx_i x_j x_k \mid ij = k \pmod{5}\}$ with i, j, k ranging over $0, 1, 2, 3, 4$.

Another solution is to let A be a sentence describing a field consisting of 5 elements. Namely, let A be the conjunction of the field axioms plus “there exist exactly 5 things”. We are using the algebraic fact that, up to isomorphism, there is exactly one field of 5 elements.

2. Let G be a group. For $a \in G$ write $a^n = a \cdots a$ (n times). We say that G is *torsion-free* if for all $a \in G$, if $a \neq 1$ then $a^n \neq 1$ for all positive integers n .
 - (a) Exhibit an infinite set of sentences, S , such that for all groups G , G is torsion-free if and only if G satisfies S .

Solution. Let $S = \{A_n : n \geq 2\}$, where A_n is the sentence

$$\forall x (x \neq 1 \Rightarrow x^n \neq 1).$$

Clearly the groups satisfying S are exactly the torsion-free groups.

(b) Show that there is no finite set S with the above property.

Solution. Suppose S' were a finite of sentences such that the groups satisfying S' are exactly the torsion-free groups. In particular, each sentence in S' is a logical consequence of the group axioms plus $S = \{A_n : n \geq 2\}$ as above. By the Compactness Theorem, each sentence in S' is a logical consequence of the group axioms plus $\{A_2, \dots, A_n\}$ for sufficiently large n . Since S' is finite, there is a fixed n such that all of the sentences in S' are logical consequences of the group axioms plus $\{A_2, \dots, A_n\}$. Now let G be a torsion group satisfying $\{A_2, \dots, A_n\}$. (For example, we may take G to be the additive group of integers modulo p , where p is a prime number greater than n .) Then G satisfies S' yet is not torsion-free, contradicting our assumption on S' .

3. Let A be the logically valid sentence $\exists x (Px \Rightarrow \forall y Py)$. Find a companion sequence C_1, \dots, C_n for A such that $(C_1 \wedge \dots \wedge C_n) \Rightarrow A$ is a quasitautology.

Solution. The unsigned tableau

$$\begin{array}{c} \neg \exists x (Px \Rightarrow \forall y Py) \\ \neg (Pa \Rightarrow \forall y Py) \\ Pa \\ \neg \forall y Py \\ \neg Pb \\ \neg (Pb \Rightarrow \forall y Py) \\ Pb \end{array}$$

is closed and shows that A is logically valid. From this tableau we read off the companion sequence

$$\begin{array}{l} C_1: (Pa \Rightarrow \forall y Py) \Rightarrow \exists x (Px \Rightarrow \forall y Py), \\ C_2: Pb \Rightarrow \forall y Py, \\ C_3: (Pb \Rightarrow \forall y Py) \Rightarrow \exists x (Px \Rightarrow \forall y Py). \end{array}$$

A simpler companion sequence for A is

$$\begin{array}{l} C'_1: Pa \Rightarrow \forall y Py, \\ C'_2: (Pa \Rightarrow \forall y Py) \Rightarrow \exists x (Px \Rightarrow \forall y Py). \end{array}$$

4. Let A be the logically valid sentence $\exists x (Px \Rightarrow \forall y Py)$. Construct either a Hilbert-style proof of A or a Gentzen-style proof of A .

Solution.

A Hilbert-style proof of A is:

1. $(Pa \Rightarrow \forall y Py) \Rightarrow \exists x (Px \Rightarrow \forall y Py)$ (EI)
2. $(\neg \exists x (Px \Rightarrow \forall y Py)) \Rightarrow Pa$ (1,QT)
3. $(\neg \exists x (Px \Rightarrow \forall y Py)) \Rightarrow \forall y Py$ (2,UG)
4. $\exists x (Px \Rightarrow \forall y Py)$ (1,3,QT).

A Gentzen-style proof of A is:

1. $Pa \rightarrow Pa$
2. $\rightarrow Pa, Pa \Rightarrow \forall y Py$ (from 1)
3. $\rightarrow Pa, \exists x (Px \Rightarrow \forall y Py)$ (from 2)
4. $\rightarrow \forall y Py, \exists x (Px \Rightarrow \forall y Py)$ (from 3)
5. $\rightarrow Pa \Rightarrow \forall y Py, \exists x (Px \Rightarrow \forall y Py)$ (from 4)
6. $\rightarrow \exists x (Px \Rightarrow \forall y Py)$ (from 5).