

# Math 557 – Midterm Exam #2

November 4, 2005

## SOLUTIONS

1. Let  $M = (U_M, f_M, g_M, I_M)$  where  $U_M = \{0, 1, 2, 3, 4\}$ ,  $I_M$  is the identity relation on  $U_M$ , and  $f_M, g_M$  are the binary operations of addition and multiplication modulo 5. Thus  $M$  is essentially just the ring of integers modulo 5. Let  $L$  be the language consisting of  $f, g, I$ . Note that  $M$  is a normal  $L$ -structure.

Write an  $L$ -sentence  $A$  such that for all normal  $L$ -structures  $M'$ ,  $M'$  satisfies  $A$  if and only if  $M'$  is isomorphic to  $M$ .

*Solution.* A brute force solution is to let  $A$  be  $\exists x_0 \exists x_1 \exists x_2 \exists x_3 \exists x_4 B$ , where  $B$  is the conjunction of  $\{\forall y (Ix_0y \vee Ix_1y \vee Ix_2y \vee Ix_3y \vee Ix_4y)\} \cup \{\neg Ix_i x_j : i \neq j\} \cup \{Ifx_i x_j x_k \mid i + j = k \pmod{5}\} \cup \{Igx_i x_j x_k \mid ij = k \pmod{5}\}$  with  $i, j, k$  ranging over  $0, 1, 2, 3, 4$ .

Another solution is to let  $A$  be a sentence describing a field consisting of 5 elements. Namely, let  $A$  be the conjunction of the field axioms plus “there exist exactly 5 things”. We are using the algebraic fact that, up to isomorphism, there is exactly one field of 5 elements.

2. Let  $G$  be a group. For  $a \in G$  write  $a^n = a \cdots a$  ( $n$  times). We say that  $G$  is *torsion-free* if for all  $a \in G$ , if  $a \neq 1$  then  $a^n \neq 1$  for all positive integers  $n$ .
  - (a) Exhibit an infinite set of sentences,  $S$ , such that for all groups  $G$ ,  $G$  is torsion-free if and only if  $G$  satisfies  $S$ .

*Solution.* Let  $S = \{A_n : n \geq 2\}$ , where  $A_n$  is the sentence

$$\forall x (x \neq 1 \Rightarrow x^n \neq 1).$$

Clearly the groups satisfying  $S$  are exactly the torsion-free groups.

(b) Show that there is no finite set  $S$  with the above property.

*Solution.* Suppose  $S'$  were a finite of sentences such that the groups satisfying  $S'$  are exactly the torsion-free groups. In particular, each sentence in  $S'$  is a logical consequence of the group axioms plus  $S = \{A_n : n \geq 2\}$  as above. By the Compactness Theorem, each sentence in  $S'$  is a logical consequence of the group axioms plus  $\{A_2, \dots, A_n\}$  for sufficiently large  $n$ . Since  $S'$  is finite, there is a fixed  $n$  such that all of the sentences in  $S'$  are logical consequences of the group axioms plus  $\{A_2, \dots, A_n\}$ . Now let  $G$  be a torsion group satisfying  $\{A_2, \dots, A_n\}$ . (For example, we may take  $G$  to be the additive group of integers modulo  $p$ , where  $p$  is a prime number greater than  $n$ .) Then  $G$  satisfies  $S'$  yet is not torsion-free, contradicting our assumption on  $S'$ .

3. Let  $A$  be the logically valid sentence  $\exists x (Px \Rightarrow \forall y Py)$ . Find a companion sequence  $C_1, \dots, C_n$  for  $A$  such that  $(C_1 \wedge \dots \wedge C_n) \Rightarrow A$  is a quasitautology.

*Solution.* The unsigned tableau

$$\begin{array}{c} \neg \exists x (Px \Rightarrow \forall y Py) \\ \neg (Pa \Rightarrow \forall y Py) \\ Pa \\ \neg \forall y Py \\ \neg Pb \\ \neg (Pb \Rightarrow \forall y Py) \\ Pb \end{array}$$

is closed and shows that  $A$  is logically valid. From this tableau we read off the companion sequence

$$\begin{array}{l} C_1: (Pa \Rightarrow \forall y Py) \Rightarrow \exists x (Px \Rightarrow \forall y Py), \\ C_2: Pb \Rightarrow \forall y Py, \\ C_3: (Pb \Rightarrow \forall y Py) \Rightarrow \exists x (Px \Rightarrow \forall y Py). \end{array}$$

A simpler companion sequence for  $A$  is

$$\begin{array}{l} C'_1: Pa \Rightarrow \forall y Py, \\ C'_2: (Pa \Rightarrow \forall y Py) \Rightarrow \exists x (Px \Rightarrow \forall y Py). \end{array}$$

4. Let  $A$  be the logically valid sentence  $\exists x (Px \Rightarrow \forall y Py)$ . Construct either a Hilbert-style proof of  $A$  or a Gentzen-style proof of  $A$ .

*Solution.*

A Hilbert-style proof of  $A$  is:

1.  $(Pa \Rightarrow \forall y Py) \Rightarrow \exists x (Px \Rightarrow \forall y Py)$  (EI)
2.  $(\neg \exists x (Px \Rightarrow \forall y Py)) \Rightarrow Pa$  (1,QT)
3.  $(\neg \exists x (Px \Rightarrow \forall y Py)) \Rightarrow \forall y Py$  (2,UG)
4.  $\exists x (Px \Rightarrow \forall y Py)$  (1,3,QT).

A Gentzen-style proof of  $A$  is:

1.  $Pa \rightarrow Pa$
2.  $\rightarrow Pa, Pa \Rightarrow \forall y Py$  (from 1)
3.  $\rightarrow Pa, \exists x (Px \Rightarrow \forall y Py)$  (from 2)
4.  $\rightarrow \forall y Py, \exists x (Px \Rightarrow \forall y Py)$  (from 3)
5.  $\rightarrow Pa \Rightarrow \forall y Py, \exists x (Px \Rightarrow \forall y Py)$  (from 4)
6.  $\rightarrow \exists x (Px \Rightarrow \forall y Py)$  (from 5).