Math 557 – Homework #4

October 24, 2005

SOLUTIONS

1. Let $L = \{R, \ldots\}$ be a language which includes a binary predicate R. Let S be a set of L-sentences. Assume that for each $n \geq 1$ there exists an L-structure $(U_n, R_n, \ldots)$ satisfying S and containing elements $a_{n1}, \ldots, a_{nn}$ such that $\langle a_{ni}, a_{nj} \rangle \in R_n$ for all i and j with $1 \leq i < j \leq n$. Prove that there there exists an L-structure $(U_\infty, R_\infty, \ldots)$ satisfying S and containing elements $a_{\infty i}$, $i = 1, 2, \ldots$ such that $\langle a_{\infty i}, a_{\infty j} \rangle \in R_\infty$ for all i and j with $1 \leq i < j$.

   Solution. Let $L^* = L \cup \{P_1, P_2, \ldots\}$ where $P_1, P_2, \ldots$ are new unary predicates. Let S^* be S plus $\exists x\, P_i x$ plus $\forall x\, \forall y\, ((P_i x \wedge P_j y) \Rightarrow Rxy)$, $1 \leq i < j$. Consider the L^*-structures $(U_n, R_n, \ldots, P_{n1}, \ldots, P_{nn}, \ldots)$, $n \geq 1$, where $P_{ni} = \{a_{ni}\}$ for $1 \leq i \leq n$, and $P_{ni} = \{\}$ for $i > n$. Clearly each finite subset of S^* is satisfied by all but finitely many of these structures. It follows by the Compactness Theorem that S^* is satisfiable. Let $(U_\infty, R_\infty, \ldots, P_{\infty 1}, P_{\infty 2}, \ldots)$ be an L^*-structure satisfying S^*. Clearly the L-structure $(U_\infty, R_\infty, \ldots)$ has the desired properties.

2. Let S be an infinite set of L-sentences, and let B be an L-sentence. Prove that $S \models B$ (i.e., B is true in all L-structures satisfying S) if and only if there exists a finite set of L-sentences $A_1, \ldots, A_k \in S$ such that $A_1, \ldots, A_k \vdash B$ (i.e., B is provable from $A_1, \ldots, A_k$).

   Solution. The "if" part follows from the soundness of our proof system. For the "only if" part, assume that $S \models B$, i.e., $S \cup \{\neg B\}$ is not satisfiable. It follows by the Compactness Theorem that there exists a finite set $\{A_1, \ldots, A_k\} \subset S$ such that $A_1, \ldots, A_k, \neg B$ is not satisfiable. Thus B is a logical consequence of $A_1, \ldots, A_k$. It follows by completeness of our proof system that $A_1, \ldots, A_k \vdash B$.

3. Write out all the axioms and rules of the Hilbert-style system LH.

   Solution.

   (a) all quasitautologies

   (b) $A,\ A \Rightarrow B\ /\ B$ (modus ponens)

   Note: We write QT to indicate *quasitautological consequence*, i.e., a quasitautology plus one or more applications of modus ponens. This is justified by Lemma 3.3.5 in the lecture notes.

   (c) $(\forall x\, A) \Rightarrow A[x/a]$ (UI)

   (d) $A[x/a] \Rightarrow (\exists x\, A)$ (EI)

   (e) $A \Rightarrow B[x/a]\ /\ A \Rightarrow (\forall x\, B)$ where a does not occur in A or B (UG)

   (f) $B[x/a] \Rightarrow A\ /\ (\exists x\, B) \Rightarrow A$ where a does not occur in A or B (EG)

4. Construct a Hilbert-style proof of the sentence

$$(\exists x\, \forall y\, Rxy) \Rightarrow (\forall y\, \exists x\, Rxy).$$

   Solution.

   1. $(\forall y\, Ray) \Rightarrow Rab$ (UI)
   2. $Rab \Rightarrow (\exists x\, Rxb)$ (EI)
   3. $(\forall y\, Ray) \Rightarrow (\exists x\, Rxb)$ (1,2,QT)
   4. $(\forall y\, Ray) \Rightarrow (\forall y\, \exists x\, Rxy)$ (3,UG)
   5. $(\exists x\, \forall y\, Rxy) \Rightarrow (\forall y\, \exists x\, Rxy)$ (4,EG)

5. Construct a Hilbert-style proof of the sentence

$$\neg\, \exists x\, (Sx \wedge \forall y\, (Eyx \Leftrightarrow (Sy \wedge \neg\, Eyy))).$$

   Solution.

   1. $(\forall y\, (Eya \Leftrightarrow (Sy \wedge \neg\, Eyy))) \Rightarrow (Eaa \Leftrightarrow (Sa \wedge \neg\, Eaa))$ (UI)
   2. $(Sa \wedge (\forall y\, (Eya \Leftrightarrow (Sy \wedge \neg\, Eyy)))) \Rightarrow (Sb \wedge \neg\, Sb)$ (1,QT)
   3. $(\exists x\, (Sx \wedge (\forall y\, (Eyx \Leftrightarrow (Sy \wedge \neg\, Eyy))))) \Rightarrow (Sb \wedge \neg\, Sb)$ (2,EG)
   4. $\neg\, \exists x\, (Sx \wedge (\forall y\, (Eyx \Leftrightarrow (Sy \wedge \neg\, Eyy))))$ (3,QT)

6. Write out all the axioms and rules of the Gentzen-style system LG.

   Solution. Left to the student.

7. Construct a Gentzen-style proof of the sequent

$$\exists x\, \forall y\, Rxy \to \forall y\, \exists x\, Rxy.$$

   Solution. A proof in LG is:

   1. $\exists x\, \forall y\, Rxy, \forall y\, Ray, Rab \to Rab, \exists x\, Rxb, \forall y\, \exists x\, Rxy$
   2. $\exists x\, \forall y\, Rxy, \forall y\, Ray, Rab \to \exists x\, Rxb, \forall y\, \exists x\, Rxy$
   3. $\exists x\, \forall y\, Rxy, \forall y\, Ray \to \exists x\, Rxb, \forall y\, \exists x\, Rxy$
   4. $\exists x\, \forall y\, Rxy \to \exists x\, Rxb, \forall y\, \exists x\, Rxy$
   5. $\exists x\, \forall y\, Rxy \to \forall y\, \exists x\, Rxy$

   To ease the writing of these Gentzen-style proofs, let LG^+ be LG augmented with the so-called *weakening rules* or *padding rules*:

$$\frac{\Gamma \to \Delta}{\Gamma, A \to \Delta} \qquad \qquad \frac{\Gamma \to \Delta}{\Gamma \to A, \Delta}$$

   Clearly LG^+ is sound and complete. Moreover, any proof in LG^+ may be straightforwardly "padded out" to a proof in LG, and any proof in LG may be rewritten as a proof in LG^+. For example, patterned on the above proof in LG, we have the following proof in LG^+:

   1. $Rab \to Rab$

   1.5. $\forall y\, Ray, Rab \to Rab$

   2. $\forall y\, Ray \to Rab$

   2.5. $\forall y\, Ray \to Rab, \exists x\, Rxb$

   3. $\forall y\, Ray \to \exists x\, Rxb$

   3.5. $\exists x\, \forall y\, Rxy, \forall y\, Ray \to \exists x\, Rxb$

   4. $\exists x\, \forall y\, Rxy \to \exists x\, Rxb$

   4.5. $\exists x\, \forall y\, Rxy \to \exists x\, Rxb, \forall y\, \exists x\, Rxy$

   5. $\exists x\, \forall y\, Rxy \to \forall y\, \exists x\, Rxy$

or, omitting the applications of the padding rules,

1. $Rab \to Rab$
2. $\forall y\, Ray \to Rab$
3. $\forall y\, Ray \to \exists x\, Rxb$
4. $\exists x\, \forall y\, Rxy \to \exists x\, Rxb$
5. $\exists x\, \forall y\, Rxy \to \forall y\, \exists x\, Rxy$

8. Construct a Gentzen-style proof of the sequent

$$\to (\exists x\, \forall y\, Rxy) \Rightarrow (\forall y\, \exists x\, Rxy).$$

Solution. Take the previous proof followed by

5.5. $\exists x\, \forall y\, Rxy \to \forall y\, \exists x\, Rxy, (\exists x\, \forall y\, Rxy) \Rightarrow (\forall y\, \exists x\, Rxy)$

6. $\to (\exists x\, \forall y\, Rxy) \Rightarrow (\forall y\, \exists x\, Rxy)$

9. Construct a Gentzen-style proof of the sequent

$$\to \neg\, \exists x\, (Sx \wedge \forall y\, (Eyx \Leftrightarrow (Sy \wedge \neg\, Eyy))).$$

Solution. In LG^+, omitting applications of the padding rules, we have:

1. $Eaa \to Eaa$ (axiom)
2. $Eaa, \neg\, Eaa \to$ (from 1)
3. $Eaa, Sa \wedge \neg\, Eaa \to$ (from 2)
4. $Sa \to Sa$ (axiom)
5. $\to Eaa, \neg\, Eaa$ (from 1)
6. $Sa \to Eaa, Sa \wedge \neg\, Eaa$ (from 4 and 5)
7. $Sa, Eaa \Leftrightarrow (Sa \wedge \neg\, Eaa) \to$ (from 4 and 6)
8. $Sa, \forall y\, (Eya \Leftrightarrow (Sy \wedge \neg\, Eyy)) \to$ (from 7)
9. $Sa \wedge \forall y\, (Eya \Leftrightarrow (Sy \wedge \neg\, Eyy)) \to$ (from 8)
10. $\exists x\, (Sx \wedge \forall y\, (Eyx \Leftrightarrow (Sy \wedge \neg\, Eyy)) \to$ (from 9)
11. $\to \neg\, \exists x\, (Sx \wedge \forall y\, (Eyx \Leftrightarrow (Sy \wedge \neg\, Eyy))$ (from 10)