Math 557 – Homework #7

October 22, 2003

SOLUTIONS

1. Construct LH-derivations of the following logically valid sentences. To make this easier, you may supplement the rules of LH with

$$\frac{A_1 \ \cdots \ A_k}{B}$$

whenever B is a quasitautological consequence of $A_1, \ldots, A_k$.

(a) $(\forall x\,(Px \,\&\, Qx)) \Leftrightarrow ((\forall x\, Px) \,\&\, (\forall x\, Qx))$

Solution.

1.	$(\forall x\,(Px \,\&\, Qx)) \Rightarrow (Pa \,\&\, Qa)$	UI
2.	$(\forall x\,(Px \,\&\, Qx)) \Rightarrow Pa$	1, QT
3.	$(\forall x\,(Px \,\&\, Qx)) \Rightarrow \forall x\, Px$	2, UG
4.	$(\forall x\,(Px \,\&\, Qx)) \Rightarrow Qa$	1, QT
5.	$(\forall x\,(Px \,\&\, Qx)) \Rightarrow \forall x\, Qx$	4, UG
6.	$(\forall x\, Px) \Rightarrow Pa$	UI
7.	$(\forall x\, Qx) \Rightarrow Qa$	UI
8.	$((\forall x\, Px) \,\&\, (\forall x\, Qx)) \Rightarrow (Pa \,\&\, Qa)$	6, 7, QT
9.	$((\forall x\, Px) \,\&\, (\forall x\, Qx)) \Rightarrow \forall x\,(Px \,\&\, Qx)$	8, UG
10.	$(\forall x\,(Px \,\&\, Qx)) \Leftrightarrow ((\forall x\, Px) \,\&\, (\forall x\, Qx))$	3, 5, 9, QT

(b) $(\exists x\, \forall y\, Rxy) \Rightarrow (\forall x\, \exists y\, Ryx)$

Solution.

1.	$(\forall y\, Ray) \Rightarrow Rab$	UI
2.	$Rab \Rightarrow \exists y\, Ryb$	EI
3.	$(\forall y\, Ray) \Rightarrow \exists y\, Ryb$	1, 2, QT
4.	$(\exists x\, \forall y\, Rxy) \Rightarrow \exists y\, Ryb$	3, EG
5.	$(\exists x\, \forall y\, Rxy) \Rightarrow \forall x\, \exists y\, Ryx$	3, UG

(c) $\neg \exists x \forall y (Eyx \Leftrightarrow \neg Eyy)$

Solution.

1.	$\forall y (Eya \Leftrightarrow \neg Eyy) \Rightarrow (Eaa \Leftrightarrow \neg Eaa)$	UI
2.	$\neg \forall y (Eya \Leftrightarrow \neg Eyy)$	1, QT
3.	$(\forall y (Eya \Leftrightarrow \neg Eyy)) \Rightarrow \neg \exists x \forall y (Eyx \Leftrightarrow \neg Eyy)$	2, QT
4.	$(\exists x \forall y (Eyx \Leftrightarrow \neg Eyy)) \Rightarrow \neg \exists x \forall y (Eyx \Leftrightarrow \neg Eyy)$	3, EG
5.	$\neg \exists x \forall y (Eyx \Leftrightarrow \neg Eyy)$	4, QT

2. The proof system LH'.

Consider the following proof system LH', which is a "stripped down" version of LH. The objects of LH' are L-V-sentences containing only $\forall$, $\Rightarrow$, $\neg$ (i.e., not containing $\exists$, $\Leftrightarrow$, $\&$, $\vee$). The rules of LH' are:

(a) quasitautologies

(b) $(\forall x\, B) \Rightarrow B[x/a]$

(c) $(\forall x\, (A \Rightarrow B)) \Rightarrow (A \Rightarrow \forall x\, B)$

(d) $\dfrac{A \quad A \Rightarrow B}{B}$ (modus ponens)

(e) $\dfrac{B[x/a]}{\forall x\, B}$ (generalization), where a does not occur in B.

Show that LH' is sound and complete.

Solution. In outline, the proof of soundness and completeness of LH' is the same as for LH. Define a companion of A to be any L-V-sentence of the form $(\forall x\, B) \Rightarrow B[x/a]$, or $B[x/a] \Rightarrow \forall x\, B$ where a does not occur in A, B. The Companion Theorem and its proof are as for LH.

The only new point is contained in the proof of the following lemma.

Lemma. If C is a companion of A and if $C \Rightarrow A$ is derivable in LH', then A is derivable in LH'.

Proof. Assume $C \Rightarrow A$ is derivable in LH'.

If C is of the form $(\forall x\, B) \Rightarrow B[x/a]$, then C is an axiom of LH', hence derivable, hence by Modus Ponens A is derivable.

Now suppose C is of the form $B[x/a] \Rightarrow \forall x\, B$ where a does not occur in A, B. Since $C \Rightarrow A$ is derivable, it follows by quasitautology that $(\neg A) \Rightarrow B[x/a]$ and $(\neg A) \Rightarrow \neg \forall x\, B$ are derivable. From the former plus generalization we have that $\forall x\, ((\neg A) \Rightarrow B)$ is derivable. But

$(\forall x\,((\neg\, A) \Rightarrow B)) \Rightarrow ((\neg\, A) \Rightarrow \forall x\, B)$ is an axiom of LH', so by Modus Ponens we have that $(\neg\, A) \Rightarrow \forall x\, B$ is derivable. Now by quasitautology we see that A is derivable. This completes the proof.

3. The proof system $LH(S)$.

   (a) Let S be a set of L-sentences. Consider a proof system $LH(S)$ consisting of LH with additional rules of inference $\langle A \rangle$, $A \in S$. Show that an L-V-sentence B is derivable in $LH(S)$ if and only if B is a logical consequence of S.

   Solution. If B is derivable in $LH(S)$, then by a straightforward induction on the length of a derivation of B in $LH(S)$, it follows that B is a logical consequence of S.

   For the converse, assume B is a logical consequence of S. By the Compactness Theorem, there exist $A_1, \ldots, A_n \in S$ such that B is a logical consequence of $A_1, \ldots, A_n$. It follows that

   $$A_1 \Rightarrow (A_2 \Rightarrow \cdots (A_n \Rightarrow B))$$

   is logically valid, hence derivable in LH (using completeness of LH), hence derivable in $LH(S)$, since $LH(S)$ includes LH. But $A_1, \ldots, A_n$ are axioms of $LH(S)$, so by n application of Modus Ponens we see that B is derivable in $LH(S)$. This completes the proof.

   (b) Indicate the modifications needed when S is a set of L-V-sentences.

   Solution. Let V' be a countably infinite set of new parameters, with $V' \cap V = \emptyset$. The objects of $LH(S)$ are L-$V \cup V'$-sentences. The rules of $LH(S)$ are as before except that, in the generalization rules, we stipulate that a does not occur in A, B, S.