Math 557 – Midterm Exam #1

October 8, 2003

SOLUTIONS

1. Find a sentence in prenex normal form which is logically equivalent to $(\forall x\, \exists y\, Rxy) \Rightarrow \neg \exists x\, Px$.

   Solution. $\exists x\, \forall y\, \forall z\, ((\neg Rxy) \vee Pz)$, or $\exists x\, \forall y\, ((\neg Rxy) \vee Py)$.

2. Using the predicates Bx ("x is a barber in Podunk") and Sxy ("x shaves y"), translate the following argument into a sentence of the predicate calculus.

   > Any barber in Podunk shaves exactly those individuals who do not shave themselves. Therefore, there is no barber in Podunk.

   Use an unsigned tableau to test this argument for logical validity.

   Solution. $(\forall x\, (Bx \Rightarrow \forall y\, (Sxy \Leftrightarrow \neg Syy))) \Rightarrow \neg \exists x\, Bx$.
   A tableau starting with the negation of this sentence (left to the student) closes off to show that the sentence is logically valid.

3. Using the predicate Rxy ("x is an ancestor of y"), translate the following argument into a sentence of the predicate calculus.

   > Every ancestor of an ancestor of an individual is an ancestor of the same individual. No individual is his own ancestor. Therefore, there is an individual who has no ancestor.

   Is this argument valid? Justify your answer by means of an appropriate structure or tableau.

   Solution. $((\forall x\, \forall y\, ((\exists z\, (Rxz \,\&\, Rzy)) \Rightarrow Rxy)) \,\&\, \neg \exists x\, Rxx) \Rightarrow \exists x\, \neg \exists y\, Ryx$.
   A tableau starting with the negation of this sentence (left to the student) fails to close off. The structure $(\mathbb{N}, >)$ falsifies the sentence, thus showing that it is not logically valid.

4. Let G be a group. The *order* of an element $a \in G$ is the smallest positive integer n such that $a^n = e$. Here e denotes the identity element of G, and

$$a^n = \underbrace{a \cdot \cdots \cdot a}_{n \text{ times}} .$$

Using only the predicates $Pxyz$ ("$x \cdot y = z$") and Ixy ("$x = y$"), write a sentence S of the predicate calculus such that, for any group G, G satisfies S if and only if G has no elements of order 2 or 3.

Solution. $\neg \exists x \, \exists y \, (Pxxx \,\&\, (\neg \, Pyyy) \,\&\, (Pyyx \vee \exists z \, (Pyyz \,\&\, Pzyx)))$.