Math 557 – Homework #5

September 22, 2003

SOLUTIONS

1. Use signed tableaux to show that the following are logically valid.

   (a) $(\forall x\,(A \Rightarrow B)) \Rightarrow ((\forall x\, A) \Rightarrow (\forall x\, B))$

$$\text{F } (\forall x\,(A \Rightarrow B)) \Rightarrow ((\forall x\, A) \Rightarrow (\forall x\, B))$$
$$\text{T } \forall x(A \Rightarrow B)$$
$$\text{F } (\forall x\, A) \Rightarrow (\forall x\, B)$$
$$\text{T } \forall x\, A$$
$$\text{F } \forall x\, B$$
$$\text{F } B[x/a]$$
$$\text{T } A[x/a]$$
$$\text{T } (A \Rightarrow B)[x/a]$$

/	\
$\text{F } A[x/a]$	$\text{T } B[x/a]$

   (b) $(\exists x\,(A \vee B)) \Leftrightarrow ((\exists x\, A) \vee (\exists x\, B))$

$$\text{F } (\exists x\,(A \vee B)) \Leftrightarrow ((\exists x\, A) \vee (\exists x\, B))$$

/	\
$\text{T } \exists x\,(A \vee B)$	$\text{F } \exists x\,(A \vee B)$
$\text{F } (\exists x\, A) \vee (\exists x\, B)$	$\text{T } (\exists x\, A) \vee (\exists x\, B)$
$\text{T } (A \vee B)[x/a]$	
$\text{F } \exists x\, A$	
$\text{F } \exists x\, B$	
$\text{F } A[x/a]$	
$\text{F } B[x/a]$	

Left branch splits:

/	\
$\text{T } A[x/a]$	$\text{T } B[x/a]$

Right branch splits:

/	\
$\text{T } \exists x\, A$	$\text{T } \exists x\, B$
$\text{T } A[x/a]$	$\text{T } B[x/a]$
$\text{F } (A \vee B)[x/a]$	$\text{F } (A \vee B)[x/a]$
$\text{F } A[x/a]$	$\text{F } A[x/a]$
$\text{F } B[x/a]$	$\text{F } B[x/a]$

(c) $(\exists x\, A) \Leftrightarrow (\neg\, \forall x \,\neg\, A)$

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\mathrm{F}\ (\exists x\, A) \Leftrightarrow (\neg\, \forall x \,\neg\, A)} \\
/ & & \backslash \\
\mathrm{T}\ \exists x\, A & & \mathrm{F}\ \exists x\, A \\
\mathrm{F}\ \neg\, \forall x \,\neg\, A & & \mathrm{T}\ \neg\, \forall x \,\neg\, A \\
\mathrm{T}\ A[x/a] & & \mathrm{F}\ \forall x \,\neg\, A \\
\mathrm{T}\ \forall x \,\neg\, A & & \mathrm{F}\ (\neg\, A)[x/a] \\
\mathrm{T}\ (\neg\, A)[x/a] & & \mathrm{T}\ A[x/a] \\
\mathrm{F}\ A[x/a] & & \mathrm{F}\ A[x/a]
\end{array}
$$

(d) $(\forall x\, (A \vee C)) \Leftrightarrow ((\forall x\, A) \vee C)$, provided x is not free in C

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\mathrm{F}\ (\forall x\, (A \vee C)) \Leftrightarrow ((\forall x\, A) \vee C)} \\
/ & & \backslash \\
\mathrm{T}\ \forall x\, (A \vee C) & & \mathrm{F}\ \forall x\, (A \vee C) \\
\mathrm{F}\ (\forall x\, A) \vee C & & \mathrm{T}\ (\forall x\, A) \vee C \\
\mathrm{F}\ \forall x\, A & & \mathrm{F}\ (A \vee C)[x/a] \\
\mathrm{F}\ C & & \mathrm{F}\ A[x/a] \\
\mathrm{F}\ A[x/a] & & \mathrm{F}\ C \\
\mathrm{T}\ (A \vee C)[x/a] & & \begin{array}{cc} / & \backslash \end{array} \\
\mathrm{T}\ A[x/a] & & \begin{array}{cc} \mathrm{T}\ \forall x\, A & \mathrm{T}\ C \end{array} \\
\mathrm{T}\ C & & \begin{array}{cc} \mathrm{T}\ A[x/a] & \end{array}
\end{array}
$$

The solution to Problem 2 is on the next page.

2. Let L be the language consisting of one binary predicate, R. Recall that an L-*structure*, M, consists of a nonempty set $U_M = U$, the *domain* or *universe* of M, together with a binary relation $R_M = R \subseteq U \times U$. A set of L-sentences S is said to be *satisfiable* if there exists an L-structure M such that all of the sentences of S are true in M.

   Consider the following sentences:

   (a) $\forall x\, Rxx$

   (b) $\forall x \neg Rxx$

   (c) $\forall x\, \forall y\, (Rxy \Rightarrow Ryx)$

   (d) $\forall x\, \forall y\, (Rxy \Rightarrow \neg Ryx)$

   (e) $\forall x\, \forall y\, \forall z\, ((Rxy \,\&\, Ryz) \Rightarrow Rxz)$

   (f) $\forall x\, \exists y\, Rxy$

   Which of subsets of this set of sentences are satisfiable? Verify your claims by exhibiting appropriate structures. Use the simplest possible structures.

   Solution.

   (a,c,e,f) is satisfiable: $U = \{1\}$, $R = \{\langle 1,1 \rangle\}$.

   (b,c,d,e) is satisfiable: $U = \{1\}$, $R = \{\,\}$.

   (b,c,f) is satisfiable: $U = \{1,2\}$, $R = \{\langle 1,2 \rangle, \langle 2,1 \rangle\}$.

   (b,d,e,f) is satisfiable: $U = \{1,2,3,\ldots\}$, $R = <$.

   These are the only maximal satisfiable sets, because:

   (a,b) is not satisfiable.

   (a,d) is not satisfiable.

   (b,c,e,f) is not satisfiable.

   (c,d,f) is not satisfiable.

   Note: (d,e,f) is not satisfiable in any finite domain.