Math 557 – Homework #6

October 15, 2003

1. Using the predicates Sx ("x is a set") and Exy ("x is a member of y"), translate the following into a sentence of the predicate calculus.

   > There exists a set whose members are exactly those sets which are not members of themselves.

   Use a tableau to test your sentence for *consistency*, i.e., satisfiability.

2. Using the predicates Sx ("x is Socrates"), Hx ("x is a man"), Mx ("x is mortal"), translate the following argument into a sentence of the predicate calculus.

   > Socrates is a man. All men are mortal. Therefore, Socrates is mortal.

   Use a tableau to test whether the argument is valid.

3. Using the predicates Sx ("x can solve this problem"), Mx ("x is a mathematician"), Jx ("x is Joe"), translate the following argument into a sentence of the predicate calculus.

   > If anyone can solve this problem, some mathematician can solve it. Joe is a mathematician and cannot solve it. Therefore, nobody can solve it.

   Use a tableau to test whether the argument is valid.

4. Using the same predicates as above, translate the following argument into a sentence of the predicate calculus.

   > Any mathematician can solve this problem if anyone can. Joe is a mathematician and cannot solve it. Therefore, nobody can solve it.

   Use a tableau to test whether the argument is valid.

5. Two L-structures are said to be *elementarily equivalent* if they satisfy the same L-sentences.

Assume that L contains a binary predicate R. Prove that, for any L-structure $M = (U_M, R_M, \ldots)$ such that (U_M, R_M) is an infinite linear ordering, there exists an L-structure $M' = (U_{M'}, R_{M'}, \ldots)$ such that

(a) M' is elementarily equivalent to M,

(b) $(U_{M'}, R_{M'})$ is a linear ordering,

(c) $(U_{M'}, R_{M'})$ contains an infinite descending sequence.

Hint: Let L^* be the language consisting of L plus additional unary predicates P_n, $n = 1, 2, \ldots$. Let S^* be the set of L^*-sentences consisting of all L-sentences true in M plus $\forall x \, \forall y \, ((P_n x \,\&\, Rxy) \Rightarrow P_n y)$, $\forall x \, (P_n x \Rightarrow P_{n+1} x)$, $\exists x \, (P_{n+1} x \,\&\, \neg \, P_n x)$, $n = 1, 2, \ldots$. Use the Compactness Theorem to show that S^* is satisfiable. Letting M^* be an L^*-structure satisfying S^*, show how M^* gives rise to an M' as desired.