Math 557: Mathematical Logic

Final Exam (Takehome)

December 8, 2000
due December 13, 2000

1. Translate the following argument into predicate calculus and use appropriate methods to establish its validity or invalidity.

   Anyone who can solve all logic problems is a good student. No student can solve every logic problem. Therefore, there are logic problems that no student can solve.

2. Consider the following argument.

   The attack will succeed only if the enemy is taken by surprise or the position is weakly defended. The enemy will not be taken by surprise unless he is overconfident. The enemy will not be overconfident if the position is weakly defended. Therefore, the attack will not succeed.

   (a) Translate the argument into propositional calculus.

   (b) Use an unsigned tableau to determine whether the argument is logically valid.

3. A formula is said to be *quantifier-free* if it contains no quantifiers. A theory T is said to *admit elimination of quantifiers* if for all formulas A there exists a quantifier-free formula A^* such that $T \vdash$ the universal closure of $A \Leftrightarrow A^*$.

   Show that for any theory T there is a definitional extension of T which admits elimination of quantifiers.

4. Let T be a theory in the predicate calculus with identity. Let P be an $n+1$-ary predicate such that

   $$T \;\vdash\; \forall x_1 \cdots \forall x_n \, \exists y \, P x_1 \cdots x_n y \; .$$

   Let f be a new n-ary function symbol (i.e., n-ary operation), and let T' consist of T plus the axiom $\forall x_1 \cdots \forall x_n \, P x_1 \cdots x_n f x_1 \cdots x_n$. Show that T' is conservative over T.

5. Let $M = (\omega, +, \cdot, 0, 1, =)$ be the standard model of first order arithmetic. Show that True_M is implicitly definable over M.

6. Let $\mathbb{R} = (\mathbb{R}, +, -, \cdot, 0, 1, <, =)$ be the ordered field of real numbers. Show that the relations $y = \mathrm{e}^x$ and $y = \sin x$ are implicitly definable over $\mathbb{R}$.

7. A consistent theory T is said to be *complete* if for every sentence B in the language of T, either $T \vdash B$ or $T \vdash \neg B$. Let T be a consistent theory which is primitive recursively axiomatizable and includes PRA. Show that T is not complete.

   Hint (Rosser): Use the Self Reference Lemma to obtain a sentence B such that $\mathsf{PRA} \vdash B \Leftrightarrow A[x/\underline{\#(B)}]$, where $A[x/\underline{\#(B)}]$ says that for any proof p of B in T there exists a proof q of $\neg B$ in T such that $\#(q) < \#(p)$. Show that $T \nvdash B$ and $T \nvdash \neg B$.