

# Math 557: Mathematical Logic

## Homework #9

October 30, 2000

This assignment is due Friday November 10.

Problem 1 is new. Problems 2 through 7 are the problems marked “left to the student” in the solutions for Homework #7 (Revised) and Homework #8.

1. A *group* is an algebraic system consisting of a set  $G$  together with a binary operation  $\cdot : G^2 \rightarrow G$ , with the following properties:
  - (a) for all  $a, b \in G$  there exists a unique element  $a \cdot b \in G$  (existence and uniqueness of products)
  - (b)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for all  $a, b, c \in G$  (associative law)
  - (c) there exists an element  $e \in G$  such that  $e \cdot a = a \cdot e = a$  for all  $a \in G$  (existence of the identity element)
  - (d) for all  $a \in G$  there exists an element  $b \in G$  such that  $a \cdot b = b \cdot a = e$  (existence of inverses).

Let  $L$  be the language with an identity predicate  $I$  (not to be confused with the identity element of a group) and a 3-ary predicate  $P$ . We identify groups  $G$  as normal  $L$ -structures  $(U_G, P_G, I_G)$  where  $U_G = G$ ,  $P_G = \{\langle a, b, c \rangle \in (U_G)^3 : a \cdot b = c\}$ , and  $I_G = \{\langle a, a \rangle : a \in U_G\}$ . Among all normal  $L$ -structures, groups are characterized by the fact that they satisfy a certain finite set of  $L$ -sentences, known as the *group axioms*. The group axioms are presented in the solution of Homework #4, Problem 4.

For  $a \in G$  write  $a^n = a \cdot \dots \cdot a$  ( $n$  times). Thus  $a^1 = a$  and  $a^{n+1} = a^n \cdot a$ . We say that  $G$  is a *torsion group* if for all  $a \in G$  there exists a positive integer  $n$  such that  $a^n = e$ .

Show that there is no set of  $L$ -sentences which characterizes the torsion groups among all groups. In other words, show that there is no set of  $L$ -sentences  $S$  with the property that, for all groups  $G$ ,  $G$  satisfies  $S$  if and only if  $G$  is a torsion group.

2. Homework #7 (Revised), Problem 5.
3. Homework #7 (Revised), Problem 6(b).
4. Homework #7 (Revised), Problem 8.
5. Homework #8, Problem 2(b).
6. Homework #8, Problem 2(c).
7. Homework #8, Problem 2(d).