Math 557 – Homework #8

October 18, 2000

1. Let L be the language consisting of the identity predicate, I, and one additional binary predicate, R. A *graph* G is a normal L-structure satisfying the L-sentences $\forall x \, \forall y \, (Rxy \Leftrightarrow Ryx)$ and $\forall x \neg Rxx$. Thus

$$G = (U_G, R_G, I_G)$$

where $U_G = \{\text{vertices of } G\}$, $R_G = \{\langle u, v\rangle \in (U_G)^2 : u \text{ is adjacent to } v\}$, and $I_G = \{\langle v, v\rangle : v \in U_G\}$.

A graph G is said to be *connected* if for any pair of vertices u, v in G there exists a *path* from u to v, i.e., a finite sequence of vertices $u = v_0, v_1, \ldots, v_n = v$ such that for each $i < n$, v_i is adjacent to v_{i+1}. Here n is the *length* of the path. The *distance* between u and v is the minimum length of a path from u to v, if such a path exists, and ∞ otherwise. The *diameter* of G is the supremum of the distances between vertices of G.

   (a) For each positive integer n, construct an L-sentence C_n such that for all graphs G, G satisfies C_n if and only if G is connected of diameter $\leq n$.

   (b) Let S be a set of L-sentences. Suppose there exist connected graphs of arbitrarily large finite diameter satisfying S. Show that there exists a non-connected graph satisfying S.

   (c) In particular, show that there is no L-sentence C such that for all graphs G, G satisfies C if and only if G is connected.

2. Show that the following L-V-sentences are derivable in LH. You may freely use Lemma 3.3.5, which says that LH is closed under quasitautological consequence.

   (a) $(\forall x \, (A \Rightarrow B)) \Rightarrow ((\forall x \, A) \Rightarrow (\forall x \, B))$

   (b) $(\exists x \, (A \vee B)) \Leftrightarrow ((\exists x \, A) \vee (\exists x \, B))$

   (c) $(\exists x \, A) \Leftrightarrow (\neg \forall x \neg A)$

   (d) $(\forall x \, (A \vee C)) \Leftrightarrow ((\forall x \, A) \vee C)$

   Note: The variable x does not occur freely in C. This follows from the fact that (d) is an L-V-sentence.

3. Consider the following proof system LH'. (LH' is a stripped down version of LH.) The objects of LH' are L-V-sentences containing only $\forall$, $\Rightarrow$, F (i.e., not containing $\exists$, $\Leftrightarrow$, &, $\vee$, $\neg$, T). The rules of inference of LH' are:

(a) quasitautologies

(b) $(\forall x\, B) \Rightarrow B[x/a]$

(c) $(\forall x\, (A \Rightarrow B)) \Rightarrow (A \Rightarrow \forall x\, B)$

(d) $\dfrac{A \quad A \Rightarrow B}{B}$ (modus ponens)

(e) $\dfrac{B[x/a]}{\forall x\, B}$ (generalization), where a does not occur in B.

Show that LH' is sound and complete.

4. (a) Let S be a set of L-sentences. Consider the proof system $LH(S)$ consisting of LH with additional rules of inference $\langle A \rangle$, $A \in S$. Show that an L-V-sentence B is derivable in $LH(S)$ if and only if B is a logical consequence of S.

   (b) Indicate the modifications needed when S is a set of L-V-sentences.

   Notation: We write $S \vdash B$ to indicate that B is derivable in $LH(S)$.