Math 557 – Homework #7

October 11, 2000

1. Let L be a language consisting of a binary predicate R and some additional predicates. Let $M = (U_M, R_M, \ldots)$ be an L-structure such that (U_M, R_M) is isomorphic to $(\mathbb{N}, <_{\mathbb{N}})$. Note that M contains no infinite R-descending sequence. Show that there exists an L-structure M' such that:

   (a) M and M' satisfy the same L-sentences.

   (b) M' contains an infinite R-descending sequence. In other words, there exist elements $a'_1, a'_2, \ldots, a'_n, \ldots \in U_{M'}$ such that $\langle a'_{n+1}, a'_n \rangle \in R_{M'}$ for all $n = 1, 2, \ldots$.

   Hint: Use the Compactness Theorem.

2. Let A be a sentence of the predicate calculus with identity. The *spectrum* of A is defined to be the set of positive integers n such that A is normally satisfiable in a domain of cardinality n. A *spectrum* is a set X of positive integers, such that $X = \text{spectrum}(A)$ for some A.

   The *spectrum problem* is the problem of characterizing the spectra, among all sets of positive integers. This is a famous and apparently difficult open problem. In particular, it is unknown whether the complement of a spectrum is necessarily a spectrum.

   Some easy exercises:

   (a) Show that if X is a finite set of positive integers, then X and the complement of X are spectra.

   (b) Show that the set of even numbers is a spectrum.

   (c) Show that the set of odd numbers is a spectrum.

   (d) Show that, if r and m are positive integers, then

   $$\{n \geq 1 : n \equiv r \bmod m\}$$

   is a spectrum.

   (e) Show that if X and Y are spectra, then $X \cup Y$ and $X \cap Y$ are spectra.

3. Let A be a sentence of the predicate calculus with identity. Assume that A is normally satisfiable in arbitrarily large finite domains. (In other words, assume that the spectrum of A is infinite.) Show that A is normally satisfiable in some infinite domain.

   Hint: Use the Compactness Theorem.

4. Let A be a sentence of the predicate calculus with identity. Show that either spectrum(A) or spectrum($\neg A$) is cofinite.

5. Let L be the following language:

   $Pxyz$: $x + y = z \bmod n$

   $Qxyz$: $x \times y = z \bmod n$

   Rxy: $x < y$

   Bx: $x = 1$ (bottom)

   Tx: $x = n$ (top)

   Nxy: $x + 1 = y$

   Ixy: $x = y$ (identity predicate)

   Exhibit a sentence Z such that the finite normal L-structures M satisfying Z consist of the integers modulo n for some positive integer n, with their usual ordering. In other words, for all finite normal L-structures M, M satisfies Z if and only if $M \cong \mathbb{Z}_n$ for some n, where

   $$\mathbb{Z}_n = (U_n, P_n, Q_n, L_n, B_n, T_n, N_n, I_n)$$

   and

   $U_n = \{1, \ldots, n\}$
   $P_n = \{\langle i, j, k\rangle \in (U_n)^3 : i + j = k \bmod n\}$
   $Q_n = \{\langle i, j, k\rangle \in (U_n)^3 : i \times j = k \bmod n\}$
   $R_n = \{\langle i, j\rangle \in (U_n)^2 : i < j\}$
   $B_n = \{1\}$
   $T_n = \{n\}$
   $N_n = \{\langle i, j\rangle \in (U_n)^2 : i + 1 = j\}$
   $I_n = \{\langle i, j\rangle \in (U_M)^2 : i = j\}$

6. (a) Show that the set of squares $\{1, 4, 9, \ldots\}$ and its complement are spectra.

   (b) Show that the set of prime numbers and its complement are spectra.

   Hint: Use the result of Exercise 5 above.

7. Show that $\{2^n : n = 1, 2, 3, \ldots\}$ and its complement are spectra.

8. Let L and $\mathbb{Z}_n$ be as in Exercise 5 above. Show that there exists an infinite normal L-structure $M = \mathbb{Z}_\infty$ with the following property: for all L-sentences A, if $\mathbb{Z}_p$ satisfies A for all sufficiently large primes p, then $\mathbb{Z}_\infty$ satisfies A.

   Hint: Use the Compactness Theorem.