

Math 557 – Final Exam

December 18, 2003

8 pages

1. Let L be the language of groups. Let S be the set of L -sentences which are true in all finite groups. Define a *pseudo-finite group* to be a group which satisfies S . Note that every finite group is pseudo-finite.
Does there exist an infinite, pseudo-finite group? Prove your answer.
2. Let A be the sentence $\neg \exists x (Sx \ \& \ \forall y (Eyx \Leftrightarrow (Sy \ \& \ \neg Eyy)))$.
 - (a) Use a closed tableau to show that A is logically valid.
 - (b) Exhibit a companion sequence C_1, \dots, C_n for A such that $(C_1 \ \& \ \dots \ \& \ C_n) \Rightarrow A$ is a quasitautology.
 - (c) Exhibit a Hilbert-style derivation of A .
3. Recall that the *spectrum* of a sentence A is the set of positive integers n such that there exists a normal model of A of cardinality n .
Let X be a subset of $\{1, 2, 3, \dots\}$. Prove that if X is a spectrum then $\{n^2 \mid n \in X\}$ is a spectrum.
4. Rosser's Theorem says that, under certain hypotheses on a theory T , there exists a PRA-sentence A such that neither $T \vdash A$ nor $T \vdash \neg A$.
 - (a) State the hypotheses on T .
 - (b) Show how to construct the sentence A . You may cite the Diagonal Lemma, a.k.a., the Self Reference Lemma.
 - (c) Sketch a proof that A has the desired properties. You may cite the Derivability Conditions.
5. Show that the binary relation $\{\langle x, y \rangle \mid e^x = y\}$ is implicitly definable over the ordered field of real numbers.
6. True or False.
 - (a) If M is a structure and R is implicitly definable over M , then R is invariant under all automorphisms of M .
 - (b) If M is a finite structure and R is implicitly definable over M , then R is explicitly definable over M .
 - (c) The theory of fields of characteristic 0 is \aleph_0 -categorical.
 - (d) If T is a consistent theory with no finite models, then T is complete if and only if T is κ -categorical for some infinite cardinal κ .

- (e) If L is a countable language and M is an L -structure, then there are only countably many relations which are implicitly definable over M .
 - (f) Let T be a consistent theory with no finite models. If T is incomplete, then there are two models of T of the same cardinality which are not isomorphic.
 - (g) For each prime number p , the theory of infinite-dimensional vector spaces over an algebraically closed field of characteristic p is complete.
 - (h) For any field F , the group of nonsingular 2×2 matrices over F is interpretable in F .
 - (i) Let T be any one of the theories Z_1 , Z_2 , ZFC. Then the set of Gödel numbers of axioms of T is primitive recursive.
 - (j) The set of Gödel numbers of sentences which are true in the structure $(\mathbb{N}, +, \cdot, =)$ is primitive recursive.
7. Let G_0 be a group which has infinitely many distinct subgroups. Prove that there exists a group G such that
- (a) G is elementarily equivalent to G_0 , and
 - (b) G has a subgroup which is not explicitly definable over G .
8. Recall that we have stated and proved the Löwenheim/Skolem Theorem only for one-sorted languages. Carefully state one or more appropriate generalizations of the Löwenheim/Skolem Theorem which apply to many-sorted languages.
- (This is a somewhat open-ended question.)