Math 557: Mathematical Logic

Homework #8

October 18, 2000

SOLUTIONS

October 30, 2000

1. Let L be the language consisting of the identity predicate, I, and one additional binary predicate, R. A *graph* G is a normal L-structure satisfying the L-sentences $\forall x \, \forall y \, (Rxy \Leftrightarrow Ryx)$ and $\forall x \, \neg \, Rxx$. Thus

$$G = (U_G, R_G, I_G)$$

where $U_G = \{\text{vertices of } G\}$, $R_G = \{\langle u, v \rangle \in (U_G)^2 : u \text{ is adjacent to } v\}$, and $I_G = \{\langle v, v \rangle : v \in U_G\}$.

A graph G is said to be *connected* if for any pair of vertices u, v in G there exists a *path* from u to v, i.e., a finite sequence of vertices $u = v_0, v_1, \ldots, v_n = v$ such that for each $i < n$, v_i is adjacent to v_{i+1}. Here n is the *length* of the path. The *distance* between u and v is the minimum length of a path from u to v, if such a path exists, and ∞ otherwise. The *diameter* of G is the supremum of the distances between vertices of G.

   (a) For each positive integer n, construct an L-sentence C_n such that for all graphs G, G satisfies C_n if and only if G is connected of diameter $\leq n$.

   (b) Let S be a set of L-sentences. Suppose there exist connected graphs of arbitrarily large finite diameter satisfying S. Show that there exists a non-connected graph satisfying S.

   (c) In particular, show that there is no L-sentence C such that for all graphs G, G satisfies C if and only if G is connected.

Solution.

   (a) As C_n we may take $\forall x \, \forall y \, (A_0 \vee A_1 \vee \cdots \vee A_n)$ where A_n is the formula

$$\exists x_1 \cdots \exists x_{n-1} \, (Rxx_1 \,\&\, Rx_1x_2 \,\&\, \cdots \,\&\, Rx_{n-1}y).$$

   Intuitively, A_n says "there exists a path of length n from x to y".

(b) Let L^* be the language consisting of L plus two additional unary predicates, P and Q. Let S' consist of S together with the sentences $\forall x \, \forall y \, (Rxy \Leftrightarrow Ryx)$ and $\forall x \, \neg \, Rxx$ and $\exists x \, Px$ and $\exists y \, Qy$ and D_n, $n = 0, 1, 2, \ldots$. Here D_n is the sentence $\forall x \, \forall y \, ((Px \,\&\, Qy) \Rightarrow \neg \, A_n)$. Intuitively, D_n says "there is no path of length n from P to Q". Note that S' is a set of L^*-sentences.

We claim that S' is normally satisfiable. By the Compactness Theorem, it suffices to show that each finite subset of S' is normally satisfiable. Let S'' be a finite subset of S'. Let n'' be so large that, for all $n \geq n''$, D_n does not belong to S''. By hypothesis, there exists a connected graph of diameter $> n''$ satisfying S. Let $G = (U_G, R_G, I_G)$ be such a graph. Pick two vertices a and b of G whose distance is $> n''$. Define $G'' = (U_G, P_{G''}, Q_{G''}, R_G, I_G)$ where $P_{G''} = \{a\}$ and $Q_{G''} = \{b\}$. Thus G'' is a normal L^*-structure satisfying S''. This proves our claim.

By our claim, let $G^* = (U_{G^*}, P_{G^*}, Q_{G^*}, R_{G^*}, I_{G^*})$ be a normal L^*-structure satisfying S'. In particular, $(U_{G^*}, R_{G^*}, I_{G^*})$ is a graph and satisfies S. Also, since G^* satisfies $\exists x \, Px$ and $\exists y \, Qy$, pick vertices $a^* \in P_{G^*}$ and $b^* \in Q_{G^*}$. For each n, since G^* satisfies D_n, the distance from a^* to b^* is $> n$. Since this holds for all n, we see that there is no (finite) path from a^* to b^*, i.e., the distance from a^* to b^* is ∞. Thus the graph $(U_{G^*}, R_{G^*}, I_{G^*})$ is not connected.

(c) Assume for a contradiction that C is an L-sentence with the property that, for all graphs G, G satisfies C if and only if G is connected. In particular, by the "if" part of the assumption, there exist finite connected graphs of arbitrarily large diameter satisfying C. (For example, consider n-cycles, where n is large.) Hence, by part (b) above with S consisting of C alone, there exists a non-connected graph satisfying C. This violates the "only if" part of the assumption.

Note: Apropos the result in part (c) above. Using a more sophisticated technique known as Ehrenfeucht-Fraïsse games, one can prove the following stronger result. There is no L-sentence C such that, for all *finite* graphs G, G satisfies C if and only if C is connected.

2. Show that the following L-V-sentences are derivable in LH. You may freely use Lemma 3.3.5, which says that LH is closed under quasitautological consequence.

   (a) $(\forall x\,(A \Rightarrow B)) \Rightarrow ((\forall x\,A) \Rightarrow (\forall x\,B))$
   (b) $(\exists x\,(A \vee B)) \Leftrightarrow ((\exists x\,A) \vee (\exists x\,B))$
   (c) $(\exists x\,A) \Leftrightarrow (\neg\,\forall x\,\neg\,A)$
   (d) $(\forall x\,(A \vee C)) \Leftrightarrow ((\forall x\,A) \vee C)$

   Note: The variable x does not occur freely in C. This follows from the fact that (d) is an L-V-sentence.

Solutions.

(a) Let D be sentence $(\forall x\,(A \Rightarrow B)) \Rightarrow ((\forall x\,A) \Rightarrow (\forall x\,B))$. To obtain a derivation of D in LH, we follow the method of our proof of completeness of LH, via the Companion Theorem.

From the unsigned tableau proof of D (i.e., the obvious finite closed unsigned tableau starting with $\neg\,D$), we obtain a companion sequence C_1, C_2, C_3 for D, where

$C_1 = B[x/a] \Rightarrow (\forall x\,B)$
$C_2 = (\forall x\,A) \Rightarrow A[x/a]$
$C_3 = (\forall x\,(A \Rightarrow B)) \Rightarrow (A[x/a] \Rightarrow B[x/a])$

and the parameter a does not occur in D or in B. Thus

$$(C_1 \,\&\, C_2 \,\&\, C_3) \Rightarrow D$$

is a quasitautology. Note also that C_2 and C_3 are instances of the universal instantiation rule of LH. Thus we have the following derivation of D in LH.

i. $(C_1 \,\&\, C_2 \,\&\, C_3) \Rightarrow D$ (quasitautology)
ii. C_2 (universal instantiation)
iii. C_3 (universal instantiation)
iv. $C_1 \Rightarrow D$ (quasitautological consequence of i, ii, iii)
v. $(\neg\,D) \Rightarrow B[x/a]$ (quasitautological consequence of iv)
vi. $(\neg\,D) \Rightarrow (\neg\,\forall x\,B)$ (quasitautological consequence of iv)
vii. $(\neg\,D) \Rightarrow (\forall x\,B)$ (universal generalization applied to v, noting that a does not occur in the conclusion)
viii. D (quasitautological consequence of vi, vii)

(b) Left to the student.

(c) Left to the student.

(d) Left to the student.

3. Consider the following proof system LH'. (LH' is a stripped down version of LH.) The objects of LH' are L-V-sentences containing only $\forall$, $\Rightarrow$, F (i.e., not containing $\exists$, $\Leftrightarrow$, &, $\vee$, $\neg$, T). The rules of inference of LH' are:

   (a) quasitautologies

   (b) $(\forall x\, B) \Rightarrow B[x/a]$

   (c) $(\forall x\, (A \Rightarrow B)) \Rightarrow (A \Rightarrow \forall x\, B)$

   (d) $\dfrac{A \quad A \Rightarrow B}{B}$ (modus ponens)

   (e) $\dfrac{B[x/a]}{\forall x\, B}$ (generalization), where a does not occur in B.

   Show that LH' is sound and complete.

   Solution.

   Soundness is proved just as for LH.

   Completeness.

   Just as for the full tableau method, we can prove soundness and completeness of the restricted tableau method with $\forall$, $\Rightarrow$, F, and from this we obtain the restricted Companion Theorem. In this context there are only two kinds of companions, the ones involving $\forall$. It remains to prove the following lemma: If C is a companion of A, and if $C \Rightarrow A$ is derivable in LH', then A is derivable in LH'.

   We deal only with companions of the form $B[x/a] \Rightarrow (\forall x\, B)$. Assume that $(B[x/a] \Rightarrow (\forall x\, B)) \Rightarrow A$ is derivable in LH', where a does not occur in A, B. It follows quasitautologically that both (i) $(\neg A) \Rightarrow B[x/a]$ and (ii) $(\neg A) \Rightarrow \neg \forall x\, B$ are derivable in LH'. From (i) and the generalization rule (e) of LH', we see that $\forall x\, ((\neg A) \Rightarrow B)$ is derivable in LH'. Also, by rule (c) of LH', $(\forall x\, ((\neg A) \Rightarrow B)) \Rightarrow ((\neg A) \Rightarrow \forall x\, B)$ is derivable in LH'. Hence, by modus ponens, $(\neg A) \Rightarrow \forall x\, B$ is derivable in LH'. It follows quasitautologically from this and (ii) that A is derivable in LH'. This completes the proof.

4. (a) Let S be a set of L-sentences. Consider the proof system $LH(S)$ consisting of LH with additional rules of inference $\langle A \rangle$, $A \in S$. Show that an L-V-sentence B is derivable in $LH(S)$ if and only if B is a logical consequence of S.

   (b) Indicate the modifications needed when S is a set of L-V-sentences.

   Notation: We write $S \vdash B$ to indicate that B is derivable in $LH(S)$.

Solution.

(a) By induction on the length of derivations, it is straightforward to prove that each sentence derivable in $LH(S)$ is a logical consequence of S. The assumption that S is a set of L-sentences (not L-V-sentences) is used in the inductive steps corresponding to rules 4(a) and 4(b), universal and existential generalization, because we need to know that the parameter a does not occur in S.

Conversely, assume B is a logical consequence of S. By the Compactness Theorem, it follows that B is a logical consequence of a finite subset of S, say $A_1, \ldots, A_n$. Hence $(A_1 \,\&\, \cdots \,\&\, A_n) \Rightarrow B$ is logically valid. Hence, by completeness of LH, $(A_1 \,\&\, \cdots \,\&\, A_n) \Rightarrow B$ is derivable in LH. Since $LH(S)$ includes LH, we have that

$$(A_1 \,\&\, \cdots \,\&\, A_n) \Rightarrow B$$

is derivable in $LH(S)$. But $A_1, \ldots, A_n$ are derivable in $LH(S)$. It follows quasitautologically that B is derivable in $LH(S)$. This completes the proof.

(b) If S is a set of L-V-sentences, we need to modify our system as follows. Let V' be a countably infinite set of new parameters, disjoint from V. Define $LH(S)$ as before, but allowing parameters from $V \cup V'$. The objects are L-$V \cup V'$-sentences. In rules 4(a) and 4(b), one must impose the restriction that a does not occur in A, B, S. With this modification, everything goes through as before.