Math 557: Mathematical Logic

Homework #7 (Revised)

October 11, 2000

SOLUTIONS

October 30, 2000

1. Let L be a finite language with identity. Let M be a finite normal L-structure. Construct an L-sentence A such that, for all normal L-structures M', M' satisfies A if and only if M' is isomorphic to M.

   Solution.

   Let $a_1, \ldots, a_k$ be the elements of U_M, and let $P, \ldots, Q$ be the predicates of L. As A we may take

   $$\exists x_1 \cdots \exists x_k \, (D_P \,\&\, \cdots \,\&\, D_Q \,\&\, \forall y \, (I x_1 y \vee \cdots \vee I x_k y))$$

   where for each n-ary predicate P of L, D_P is the conjunction of clauses $P x_{i_1} \cdots x_{i_n}$ for each n-tuple $\langle a_{i_1}, \ldots, a_{i_n} \rangle \in P_M$, and $\neg P x_{i_1} \cdots x_{i_n}$ for each n-tuple $\langle a_{i_1}, \ldots, a_{i_n} \rangle \notin P_M$.

2. Let L be the following language:

   Ox: $x = 1$

   $Pxyz$: $x + y = z$

   $Qxyz$: $x \times y = z$

   Rxy: $x < y$

   Sxy: $x + 1 = y$

   Ixy: $x = y$ (identity predicate)

   For each positive integer n, let M_n be the L-structure

   $$M_n = (U_n, O_n, P_n, Q_n, R_n, S_n, I_n)$$

   where

$$U_n = \{1, \ldots, n\}$$
$$O_n = \{1\}$$
$$P_n = \{\langle i, j, k\rangle \in (U_n)^3 : i + j = k\}$$
$$Q_n = \{\langle i, j, k\rangle \in (U_n)^3 : i \times j = k\}$$
$$R_n = \{\langle i, j\rangle \in (U_n)^2 : i < j\}$$
$$S_n = \{\langle i, j\rangle \in (U_n)^2 : i + 1 = j\}$$
$$I_n = \{\langle i, j\rangle \in (U_M)^2 : i = j\}$$

Exhibit a sentence Z such that, for all finite normal L-structures M', M' satisfies Z if and only if M' is isomorphic to M_n for some n.

Solution.

As Z we may take the conjunction of the following clauses.

(a) $\forall x\, \forall y\, (Rxy \lor Ryx \lor Ixy)$

(b) $\forall x\, \forall y\, (Rxy \Rightarrow \neg\, Ryx)$

(c) $\forall x\, \forall y\, \forall z\, ((Rxy \,\&\, Ryz) \Rightarrow Rxz)$

(d) $\forall x\, \forall z\, (Sxz \Leftrightarrow (Rxz \,\&\, \neg\, \exists y\, (Rxy \,\&\, Ryz)))$

(e) $\forall u\, (Ou \Leftrightarrow \neg\, \exists x\, Rxu)$

(f) $\forall u\, (Ou \Rightarrow \forall x\, \forall z\, (Puxz \Leftrightarrow Sxz))$

(g) $\forall v\, \forall w\, (Svw \Rightarrow \forall x\, \forall z\, (Pwxz \Leftrightarrow \exists y\, (Syz \,\&\, Pvxy)))$

(h) $\forall u\, (Ou \Rightarrow \forall x\, \forall z\, (Quxz \Leftrightarrow Ixz))$

(i) $\forall v\, \forall w\, (Svw \Rightarrow \forall x\, \forall z\, (Qwxz \Leftrightarrow \exists y\, (Qvxy \,\&\, Pxyz)))$

Clauses (a), (b) and (c) say that R is an irreflexive linear ordering of the universe. Clause (d) says that S is the immediate successor relation, with respect to R. Clause (e) says that 1 is the first element of the universe, with respect to R. Clauses (f) and (g) define the addition predicate P, by induction along R, in terms of S. Clauses (h) and (i) define the multiplication predicate Q, by induction along R, in terms of S and P.

3. Let A be a sentence of the predicate calculus with identity. The *spectrum of* A is defined to be the set of positive integers n such that A is normally satisfiable in a domain of cardinality n. A *spectrum* is a set X of positive integers, such that $X = \text{spectrum}(A)$ for some A.

   The *spectrum problem* is the problem of characterizing the spectra, among all sets of positive integers. This is a famous and apparently difficult open problem. In particular, it is unknown whether the complement of a spectrum is necessarily a spectrum.

   Some easy exercises:

   (a) Show that if X is a finite or cofinite set of positive integers, then X is a spectrum. (By a cofinite set we mean the complement of a finite set.)

   (b) Show that the set of even numbers is a spectrum.

   (c) Show that the set of odd numbers is a spectrum.

   (d) Show that, if r and m are positive integers, then

   $$\{n \geq 1 : n \equiv r \bmod m\}$$

   is a spectrum.

   (e) Show that, if X and Y are spectra, then $X \cup Y$ and $X \cap Y$ are spectra.

Solutions.

(a) Let E_n be sentence in the language with only the identity predicate I, saying that the universe consists of exactly n elements. (Details of E_n are in Exercise 4.1.10 of the lecture notes.) If $X = \{n_1, \ldots, n_k\}$, then X is the spectrum of $E_{n_1} \vee \cdots \vee E_{n_k}$, and the complement of X is spectrum of

$$\neg (E_{n_1} \vee \cdots \vee E_{n_k}).$$

(b) The even numbers are the spectrum of a sentence which says: R is an equivalence relation on the universe, such that each equivalence class consists of exactly two elements. For more details, see Exercise 4.1.16 in the lecture notes.

(c) The odd numbers are the spectrum of a sentence which says: R is an equivalence relation on the universe, such that each equivalence class consists of exactly two elements, except for one equivalence class, which consists of exactly one element.

(d) We may assume that $0 \leq r < m$. If $r = 0$, the set

$$\{n \geq 1 : n \equiv 0 \bmod m\} = \{n \geq 1 : m \text{ divides } n \text{ with no remainder}\}$$

is the spectrum of a sentence which says: R is an equivalence relation on the universe, such that each equivalence class consists of exactly m elements. If $r > 0$, the set

$$\{n \geq 1 : n \equiv r \bmod m\} = \{n \geq 1 : m \text{ divides } n \text{ with remainder } r\}$$

is the spectrum of a sentence which says: R is an equivalence relation on the universe, such that each equivalence class consists of exactly m elements, except for one equivalence class, which consists of exactly r elements.

(e) Assume that X is the spectrum of A and Y is the spectrum of B. Then $X \cup Y$ is the spectrum of $A \vee B$. Also, $X \cap Y$ is the spectrum of $A \,\&\, B$, provided A and B have no predicates in common except the identity predicate. To arrange for this, replace B by an analogous sentence in a different language.

4. (a) Show that the set of prime numbers and its complement are spectra.

(b) Show that the set of squares $\{1, 4, 9, \ldots\}$ and its complement are spectra.

(c) Show that $\{2^n : n = 1, 2, 3, \ldots\}$ and its complement are spectra.

(d) Show that the set of prime powers $\{p^n : p \text{ prime}, n = 1, 2, \ldots\}$ and its complement are spectra.

Hint: Use the result of Exercise 2 above.

Solution.

Let Z be as in Exercise 2 above. For each of the given sets X, we exhibit a sentence A with the following properties: X is the spectrum of $Z \,\&\, A$, and the complement of X is the spectrum of $Z \,\&\, \neg A$.

(a) $\exists z\, ((\neg \exists w\, Rzw) \,\&\, (\neg \exists x\, \exists y\, (Rxz \,\&\, Ryz \,\&\, Qxyz)) \,\&\, (\neg Oz))$.

(b) $\exists z\, ((\neg \exists w\, Rzw) \,\&\, \exists x\, Qxxz)$.

(c) $\exists z\, \exists v\, ((\neg \exists w\, Rzw) \,\&\, (\exists u\, (Ou \,\&\, Suv))$
$\&\, \forall x\, ((\neg Ox \,\&\, \exists y\, Qxyz) \Rightarrow \exists w\, Qvwx))$.

(d) $\exists z\, \exists v\, ((\neg \exists w\, Rzw) \,\&\, (\neg \exists x\, \exists y\, (Rxv \,\&\, Ryv \,\&\, Qxyv)) \,\&\, (\neg Ov)$
$\&\, \forall x\, ((\neg Ox \,\&\, \exists y\, Qxyz) \Rightarrow \exists w\, Qvwx))$.

5. (a) The Fibonacci numbers are defined recursively by $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Show that the set of Fibonacci numbers $\{F_n : n = 1, 2, \ldots\} = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$ and its complement are spectra.

(b) Show that $\{x^y : x, y \geq 2\}$ and its complement are spectra.

Solution. Left to the student.

6. Let A be a sentence of the predicate calculus with identity.

   (a) Assume that A is normally satisfiable in arbitrarily large finite domains. (In other words, assume that the spectrum of A is infinite.) Show that A is normally satisfiable in some infinite domain.

   (b) Show that at least one of spectrum(A) and spectrum($\neg A$) is cofinite.

   Hint: Use the Compactness Theorem.

   Solution.

   (a) Let K_n be a sentence in the language with only the identity predicate I, saying that the universe contains at least n elements. Let S be a set of sentences consisting of $A, K_1, K_2, \ldots, K_n, \ldots$. Since A is normally satisfiable in arbitrarily large finite domains, it follows that each finite subset of S is normally satisfiable. Hence, by the Compactness Theorem for normal satisfiability (Corollary 4.1.9 in the lecture notes), S is normally satisfiable. Let M be a normal L-structure satisfying S. Since M satisfies K_n for each n, U_M of M is infinite. Thus A is normally satisfiable in the domain U_M. This proves the desired result.

   (b) Left to the student.

7. Let L and M_n be as in Exercise 2 above. Show that there exists an infinite normal L-structure $M = M_\infty$ with the following property: for all L-sentences A, if M_p satisfies A for all sufficiently large primes p, then M_∞ satisfies A.

   Hint: Use the Compactness Theorem.

   Solution.

   Let S be the set of L-sentences A with the following property: there exists $n = n_A$ such that for all primes $p > n_A$, M_p satisfies A.

   We claim that every finite subset of S is normally satisfiable. To see this, let $S_0 = \{A_1, \ldots, A_k\}$ be a finite subset of S. Put $n_0 = max(n_{A_1}, \ldots, n_{A_k})$. Let p be any prime $> n_0$. Then M_p satisfies $A_1, \ldots, A_k$. This proves our claim.

   By the Compactness Theorem for normal satisfiability (Corollary 4.1.9 in the lecture notes), it follows that S is normally satisfiable. Let M_∞ be a normal L-structure satisfying S. Among the sentences of S are those asserting that the universe has at least n elements, for each positive integer n. Since M_∞ satisfies these sentences, it follows that M_∞ is infinite.

8. Let L be a language consisting of a binary predicate R and some additional predicates. Let $M = (U_M, R_M, \ldots)$ be an L-structure such that (U_M, R_M) is isomorphic to $(\mathbb{N}, <_{\mathbb{N}})$. Note that M contains no infinite R-descending sequence. Show that there exists an L-structure M' such that:

   (a) M and M' satisfy the same L-sentences.

   (b) M' contains an infinite R-descending sequence. In other words, there exist elements $a'_1, a'_2, \ldots, a'_n, \ldots \in U_{M'}$ such that $\langle a'_{n+1}, a'_n \rangle \in R_{M'}$ for all $n = 1, 2, \ldots$.

   Hint: Use the Compactness Theorem.

   Solution. Left to the student.