

Math 557 – Homework #6

September 22, 2000

SOLUTIONS

1. (10 points)

Let L be the language consisting of one binary predicate, R . Recall that an L -structure, M , consists of a nonempty set $U_M = U$, the *domain* or *universe* of M , together with a binary relation $R_M = R \subseteq U \times U$. A set of L -sentences S is said to be *satisfiable* if there exists an L -structure M such that all of the sentences of S are true in M .

Consider the following sentences:

- (a) $\forall x Rxx$
- (b) $\forall x \neg Rxx$
- (c) $\forall x \forall y (Rxy \Rightarrow Ryx)$
- (d) $\forall x \forall y (Rxy \Rightarrow \neg Ryx)$
- (e) $\forall x \forall y \forall z ((Rxy \& Ryz) \Rightarrow Rxz)$
- (f) $\forall x \exists y Rxy$

Which of subsets of this set of sentences are satisfiable? Verify your claims by exhibiting appropriate structures. Use the simplest possible structures.

Solution.

- (a,c,e,f) is satisfiable: $U = \{1\}$, $R = \{\langle 1, 1 \rangle\}$.
- (b,c,d,e) is satisfiable: $U = \{1\}$, $R = \{\}$.
- (b,c,f) is satisfiable: $U = \{1, 2\}$, $R = \{\langle 1, 2 \rangle, \langle 2, 1 \rangle\}$.
- (b,d,e,f) is satisfiable: $U = \{1, 2, 3, \dots\}$, $R = <$.

These are the only maximal satisfiable sets, because:

- (a,b) is not satisfiable.
- (a,d) is not satisfiable.
- (b,c,e,f) is not satisfiable.
- (c,d,f) is not satisfiable.

Note: (d,e,f) is not satisfiable in any finite domain.

2. (10 points)

Use signed tableaux to show that the following are logically valid.

(a) $(\forall x (A \Rightarrow B)) \Rightarrow ((\forall x A) \Rightarrow (\forall x B))$

$$\begin{array}{c}
 \text{F } (\forall x (A \Rightarrow B)) \Rightarrow ((\forall x A) \Rightarrow (\forall x B)) \\
 \text{T } \forall x (A \Rightarrow B) \\
 \text{F } (\forall x A) \Rightarrow (\forall x B) \\
 \text{T } \forall x A \\
 \text{F } \forall x B \\
 \text{F } B[x/a] \\
 \text{T } A[x/a] \\
 \text{T } (A \Rightarrow B)[x/a] \\
 \begin{array}{cc}
 / & \backslash \\
 \text{F } A[x/a] & \text{T } B[x/a]
 \end{array}
 \end{array}$$

(b) $(\exists x (A \vee B)) \Leftrightarrow ((\exists x A) \vee (\exists x B))$

$$\begin{array}{c}
 \text{F } (\exists x (A \vee B)) \Leftrightarrow ((\exists x A) \vee (\exists x B)) \\
 \begin{array}{cc}
 / & \backslash \\
 \text{T } \exists x (A \vee B) & \text{F } \exists x (A \vee B) \\
 \text{F } (\exists x A) \vee (\exists x B) & \text{T } (\exists x A) \vee (\exists x B) \\
 \text{T } (A \vee B)[x/a] & / \quad \backslash \\
 \text{F } \exists x A & \text{T } \exists x A \quad \text{T } \exists x B \\
 \text{F } \exists x B & \text{T } A[x/a] \quad \text{T } B[x/a] \\
 \text{F } A[x/a] & \text{F } (A \vee B)[x/a] \quad \text{F } (A \vee B)[x/a] \\
 \text{F } B[x/a] & \text{F } A[x/a] \quad \text{F } A[x/a] \\
 / & \backslash \\
 \text{T } A[x/a] & \text{T } B[x/a] \\
 \text{F } B[x/a] & \text{F } B[x/a]
 \end{array}
 \end{array}$$

(c) $(\exists x A) \Leftrightarrow (\neg \forall x \neg A)$

$$\begin{array}{c}
 \text{F } (\exists x A) \Leftrightarrow (\neg \forall x \neg A) \\
 / \quad \backslash \\
 \text{T } \exists x A \quad \text{F } \exists x A \\
 \text{F } \neg \forall x \neg A \quad \text{T } \neg \forall x \neg A \\
 \text{T } A[x/a] \quad \text{F } \forall x \neg A \\
 \text{T } \forall x \neg A \quad \text{F } (\neg A)[x/a] \\
 \text{T } (\neg A)[x/a] \quad \text{T } A[x/a] \\
 \text{F } A[x/a] \quad \text{F } A[x/a]
 \end{array}$$

(d) $(\forall x (A \vee C)) \Leftrightarrow ((\forall x A) \vee C)$, provided x is not free in C

$$\begin{array}{ccc}
 & \text{F } (\forall x (A \vee C)) \Leftrightarrow ((\forall x A) \vee C) & \\
 & / \qquad \qquad \qquad \backslash & \\
 \text{T } \forall x (A \vee C) & & \text{F } \forall x (A \vee C) \\
 \text{F } (\forall x A) \vee C & & \text{T } (\forall x A) \vee C \\
 \text{F } \forall x A & & \text{F } (A \vee C)[x/a] \\
 \text{F } C & & \text{F } A[x/a] \\
 \text{F } A[x/a] & & \text{F } C \\
 \text{T } (A \vee C)[x/a] & & / \quad \backslash \\
 \text{T } A[x/a] & & \text{T } \forall x A \quad \text{T } C \\
 \text{T } C & & \text{T } A[x/a]
 \end{array}$$