Math 557 – Homework #5

September 13, 2000

SOLUTIONS

Exercises in writing formulas of predicate calculus.

1. Assume the following predicates:

   Hx: x is a human
   Cx: x is a car
   Tx: x is a truck
   Dxy: x drives y

   Write formulas representing the obvious assumptions: no human is a car, no car is a truck, humans exist, cars exist, only humans drive, only cars and trucks are driven, etc.

   Solution.

   No human is a car. $\neg \exists x \, (Hx \,\&\, Cx)$

   No car is a truck. $\neg \exists x \, (Cx \,\&\, Tx)$

   Humans exist. $\exists x \, Hx$

   Cars exist. $\exists x \, Cx$

   Only humans drive. $\forall x \, ((\exists y \, Dxy) \Rightarrow Hx)$

   Only cars and trucks are driven. $\forall x \, ((\exists y \, Dyx) \Rightarrow (Cx \vee Tx))$

   Some humans drive. $\exists x \, (Hx \,\&\, \exists y \, Dxy)$

   Some humans do not drive. $\exists x \, (Hx \,\&\, \neg \exists y \, Dxy)$

   Some cars are driven. $\exists x \, (Cx \,\&\, \exists y \, Dyx)$.

   Some cars are not driven (e.g., old wrecks). $\exists x \, (Cx \,\&\, \neg \exists y \, Dyx)$

   etc, etc.

2. Write formulas representing the following statements:

   (a) Everybody drives a car or a truck.
   Solution. $\forall x \, (Hx \Rightarrow \exists y \, (Dxy \,\&\, (Cy \vee Ty)))$

   (b) Some people drive both.
   Solution. $\exists x \, (Hx \,\&\, \exists y \, \exists z \, (Dxy \,\&\, Cy \,\&\, Dxz \,\&\, Tz))$

   (c) Some people don't drive either.
   Solution. $\exists x \, (Hx \,\&\, \neg \exists y \, (Dxy \,\&\, (Cy \vee Ty)))$

   (d) Nobody drives both.
   Solution. $\neg \exists x \, (Hx \,\&\, \exists y \, \exists z \, (Dxy \,\&\, Dxz \,\&\, Cy \,\&\, Tz))$

3. Assume in addition the following predicate:

   Ixy: x is identical to y

   Write formulas representing the following statements:

   (a) Every car has at most one driver.
   Solution. $\forall x\,(Cx \Rightarrow \forall y\,\forall z\,((Dyx \,\&\, Dzx) \Rightarrow Iyz))$

   (b) Every truck has exactly two drivers.
   Solution. $\forall x\,(Tx \Rightarrow \exists y\,\exists z\,((\neg\, Iyz) \,\&\, Dyx \,\&\, Dzx \,\&\, \forall w\,(Dwx \Rightarrow (Iwy \vee Iwz))))$

   (c) Everybody drives exactly one vehicle (car or truck).
   Solution. $\forall x\,(Hx \Rightarrow \exists y\,(Dxy \,\&\, (Cy \vee Ty) \,\&\, \forall z\,((Dxz \,\&\, (Cz \vee Tz)) \Rightarrow Iyz)))$

4. Assume the following predicates:

   Ixy: $x = y$

   $Pxyz$: $x \cdot y = z$

   Write formulas representing the axioms for a group: axioms for equality, existence and uniqueness of products, associative law, existence of an identity element, existence of inverses.

   Solution.

   (a) equality axioms:
      i. $\forall x\, Ixx$ [reflexivity]
      ii. $\forall x\,\forall y\,(Ixy \Leftrightarrow Iyx)$ [symmetry]
      iii. $\forall x\,\forall y\,\forall z\,((Ixy \,\&\, Iyz) \Rightarrow Ixz)$ [transitivity]
      iv. $\forall x\,\forall x'\,\forall y\,\forall y'\,\forall z\,\forall z'\,((Ixx' \,\&\, Iyy' \,\&\, Izz') \Rightarrow (Pxyz \Leftrightarrow Px'y'z'))$ [congruence with respect to P]

   (b) existence and uniqueness of products:
      i. $\forall x\,\forall y\,\exists z\, Pxyz$ [existence]
      ii. $\forall x\,\forall y\,\forall z\,\forall w\,((Pxyz \,\&\, Pxyw) \Rightarrow Izw)$ [uniqueness]

   (c) associative law:
   $\forall x\,\forall y\,\forall z\,\exists u\,\exists v\,\exists w\,(Pxyu \,\&\, Pyzv \,\&\, Puzw \,\&\, Pxvw)$

   (d) existence of identity element:
   $\exists u\,\forall x\,(Puxx \,\&\, Pxux)$

   (e) existence of inverses:
   $\exists u\,\forall x\,\exists y\,(Puxx \,\&\, Pxux \,\&\, Pxyu \,\&\, Pyxu)$