

# Math 557 – Midterm Exam #2

November 4, 2005

4 problems

1. Let  $M = (U_M, f_M, g_M, I_M)$  where  $U_M = \{0, 1, 2, 3, 4\}$ ,  $I_M$  is the identity relation on  $U_M$ , and  $f_M, g_M$  are the binary operations of addition and multiplication modulo 5. Thus  $M$  is essentially just the ring of integers modulo 5. Let  $L$  be the language consisting of  $f, g, I$ . Note that  $M$  is a normal  $L$ -structure.

Write an  $L$ -sentence  $A$  such that for all normal  $L$ -structures  $M'$ ,  $M'$  satisfies  $A$  if and only if  $M'$  is isomorphic to  $M$ .

2. Let  $G$  be a group. For  $a \in G$  write  $a^n = a \cdots a$  ( $n$  times). We say that  $G$  is *torsion-free* if for all  $a \in G$ , if  $a \neq 1$  then  $a^n \neq 1$  for all positive integers  $n$ .
  - (a) Exhibit an infinite set of sentences,  $S$ , such that for all groups  $G$ ,  $G$  is torsion-free if and only if  $G$  satisfies  $S$ .
  - (b) Show that there is no finite set  $S$  as above.
3. Let  $A$  be the logically valid sentence  $\exists x (Px \Rightarrow \forall y Py)$ . Find a companion sequence  $C_1, \dots, C_n$  for  $A$  such that  $(C_1 \wedge \cdots \wedge C_n) \Rightarrow A$  is a quasitautology.
4. Let  $A$  be the logically valid sentence  $\exists x (Px \Rightarrow \forall y Py)$ . Construct either a Hilbert-style proof of  $A$  or a Gentzen-style proof of  $A$ .