MATH 457: Mathematical Logic

Homework # 7

April 15, 2005

1. Exhibit a register machine program which computes the exponential function
$$\exp(m, n) = m^n.$$
Note: $m^0 = 1$ for all m. In particular, $0^0 = 1$. Also, $0^n = 0$ for all $n \geq 1$.

2. Assume that $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are one-place, total computable functions. Assume that $\mathcal{P}$ and $\mathcal{Q}$ are register machine programs which compute f and g respectively. Consider the one-place, total computable function h defined by putting
$$h(m) = f(g(m))$$
for all m.

   Note: h is called the *composition* of f and g. The notation $h = f \circ g$ is sometimes used.

   Show how to combine $\mathcal{P}$ and $\mathcal{Q}$ to create a register machine program which computes h.

3. Assume that $f : \mathbb{N} \xrightarrow{1-1} \mathbb{N}$ is a one-place, total computable, one-to-one function. Let $\mathcal{P}$ be a register machine program which computes f. Consider the inverse function f^{-1} defined by
$$f^{-1}(n) = \begin{cases} \text{the unique } m \text{ such that } f(m) = n, \text{ if such an } m \text{ exists,} \\ \text{undefined otherwise.} \end{cases}$$
   Note that f^{-1} is a one-place, partial computable, one-to-one function. Use $\mathcal{P}$ to create a register machine program $\mathcal{P}^{-1}$ which computes f^{-1}.

4. (a) Exhibit a register machine program $\mathcal{P}_2$ which computes the function $f(m) = 2m$.
   (b) Evaluate $k_2 = \#(\mathcal{P}_2)$, the Gödel number of $\mathcal{P}_2$.
   (c) Evaluate $\varphi(k_2, 0)$, $\varphi(k_2, 1)$, $\varphi(k_2, 2)$, $\varphi(k_2, 3)$, and $\varphi(k_2, 100)$.