Math 457: Introduction to Mathematical Logic

Final Exam

May 2, 2005

1. Find a sentence in prenex form which is logically equivalent to the following sentence:

$$(\forall x\, \exists y\, Rxy) \wedge (\forall y\, \exists x\, Rxy) \wedge (\forall x\, \exists y\, \neg Rxy) \wedge (\forall y\, \exists x\, \neg Rxy).$$

2. Assume the following predicates:

   Px: x is a point

   Lx: x is a straight line

   Exy: x lies on y

   Ixy: x is identical to y

   (a) Translate the following statement into a sentence of the predicate calculus with identity:

   For any two distinct points, there is a unique straight line on which they both lie.

   (b) Exhibit a normal structure which satisfies your sentence.

3. Use a tableau (signed or unsigned) to test the following sentence for logical validity.

$$(\forall x\, Rxx) \Rightarrow ((\forall x\, \exists y\, Rxy) \wedge (\forall y\, \exists x\, Rxy)).$$

4. Let S be the following set of sentences in the predicate calculus with identity.

   $\forall x\, \exists y\, (Rxy \wedge Py)$

   $\exists x\, \exists y\, (Rxy \wedge \neg Py)$

   $\forall x\, \forall y\, \forall z\, ((Rxy \wedge Rxz) \Rightarrow Iyz)$

   Use a tableau to test S for normal satisfiability.

   (Hint: In addition to the sentences of S, your tableau may involve one or more of the identity axioms for the appropriate language.)

5. Consider the language $L = \{R, I\}$ where R is a binary predicate and I is the identity predicate. Note that $(\mathbb{N}, <, =)$ is a normal L-structure. Prove that there is no L-sentence A such that, up to isomorphism, the only normal L-structure satisfying A is $(\mathbb{N}, <, =)$.

   (Hint: Use the Compactness Theorem.)

6. Show how to construct a sentence of the predicate calculus with identity whose spectrum is $\{n^2 + 1 \mid n \in \mathbb{N}\} = \{1, 2, 5, 10, 17, \ldots\}$.

7. Exhibit a register machine program which computes the following partial computable function:

$$\psi(x) = \begin{cases} 2x + 1 & \text{if } x \geq 2\,, \\ \uparrow & \text{if } x \leq 1\,. \end{cases}$$

8. Prove that there exist two semicomputable sets $A, B \subseteq \mathbb{N}$ such that $A \cap B = \emptyset$ yet there is no computable set C with $A \subseteq C$ and $C \cap B = \emptyset$.

   (Hint: Imitate Turing's proof that there exists a set $K \subseteq \mathbb{N}$ such that K is semicomputable but not computable.)