Math 312, Intro. to Real Analysis: Midterm Exam #2

Stephen G. Simpson

Friday, March 27, 2009

1. True or False (2 points each)
   (a) Every monotone sequence of real numbers is convergent.
   (b) Every sequence of real numbers has a $\limsup$ and a $\liminf$.
   (c) Every sequence of real numbers has a monotone subsequence.
   (d) Every sequence of real numbers has a convergent subsequence.
   (e) If $\liminf a_n = \limsup a_n = \alpha$ then $\lim a_n = \alpha$.
   (f) We can find a sequence of real numbers, (a_n), such that the subsequential limits of (a_n) are exactly the real numbers in the closed interval $[-1, 1]$.
   (g) The series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ is absolutely convergent.
   (h) If $\sum |a_n|$ is convergent, then so is $\sum |a_n|^2$.
   (i) If $\sum a_n$ is convergent, then so is $\sum a_n^2$.
   (j) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for all $p \geq 1$.
   (k) $\sum_{n=1}^{\infty} \frac{1}{10^n} = \frac{1}{9}$.
   (l) The function $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.
   (m) If a function is uniformly continuous on the interval $(a, b]$ and on the interval $[b, c)$, then it is uniformly continuous on the interval (a, c).

2. (6 points each) Which of the following series are convergent and/or absolutely convergent? Please indicate which tests you are using and show your work.
   (a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
   (b) $\sum_{n=1}^{\infty} (\sqrt{n^2 + n} - n)$
   (c) $\sum_{n=1}^{\infty} (\sqrt{n^2 + 1} - n)$
   (d) $\sum \frac{1}{1.01^n - 1000}$
   (e) $\sum \frac{1}{n^2 + \sqrt{n^2 + 1}}$
   (f) $\sum_{n=1}^{\infty} \log \frac{n+1}{n}$

3. (8 points) Use algebra plus limit laws to calculate

$$\lim \frac{\log \sqrt{e^{(2n+11)/n}}}{\sin((4n\pi + 5)/16n)}.$$

In performing this calculation, you may take it for granted that functions such as $\sin x$, $\cos x$, e^x, $\log x$, $\sqrt{x}$, etc. are continuous. Please show your work.

4. (5 points) Assume that $f(x)$ is continuous on the closed interval $[0, 1]$. Assume also that $f(x) \in [0, 1]$ for all $x \in [0, 1]$. Using known theorems about continuous functions, prove that the equation $f(x) = x$ has at least one solution in $[0, 1]$.

5. (3 points each) Which of the following functions are continous and/or uniformly continuous on the specified domain?

   (a) $1/x$ on its natural domain.

   (b) $1/x$ on $(1, \infty)$.

   (c) $x \cos \frac{1}{x}$ on $(0, 10]$.

   (d) $\sqrt{x}$ on $[0, \infty)$.

   (e) $f(x) = \begin{cases} |x|/x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$ on $(-\infty, \infty)$.

6. (10 points) It is known that the function $\sqrt{x}$ is uniformly continuous on the interval $[0.01, 100]$. Given $\epsilon > 0$, find a $\delta > 0$ (depending only on ϵ) such that $|\sqrt{x} - \sqrt{y}| < \epsilon$ whenever $0.01 \leq x < y \leq 100$ and $|x - y| < \delta$.