A MULTI-DIMENSIONAL OPTIMAL HARVESTING PROBLEM
WITH MEASURE VALUED SOLUTIONS

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Abstract. The paper is concerned with the optimal harvesting of a marine resource, described by an elliptic equation with Neumann boundary conditions and a nonlinear source term. Since the cost function has linear growth, an optimal solution is found within the class of measure-valued control strategies. The paper also provides results on the existence and uniqueness of strictly positive solutions to the elliptic equation, and an averaging inequality valid for subharmonic functions with Neumann boundary data.

1. The basic model

In this paper we study an optimal harvesting problem in a multi-dimensional domain. Consider a bounded connected open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary. Denote by $\varphi = \varphi(t, x)$ the density of fish at time $t$ at the point $x \in \Omega$. In absence of fishing activity, assume that the fish population evolves according to the parabolic equation with source term

$$\varphi_t = \Delta \varphi + g(x, \varphi) \quad x \in \Omega,$$

with Neumann boundary conditions

$$(1.1) \quad \nabla \varphi \cdot n = 0 \quad x \in \partial \Omega.$$ 

Here $n = n(x)$ denotes the unit outer normal to the set $\Omega$ at the point $x \in \partial \Omega$. A typical choice for the source term is

$$g(x, \varphi) = \alpha(x)(h(x) - \varphi) \varphi.$$

Here $h(x)$ denotes the maximum fish population that can be supported by the habitat at $x$, while $\alpha$ is a reproduction speed.

We denote by $u = u(t, x)$ the intensity of harvesting conducted by a fishing company. In the presence of this harvesting activity, the population evolves according to

$$\varphi_t = \Delta \varphi + g(x, \varphi) - \varphi u.$$

Assuming that the harvesting rate remains constant in time, the fish population will reach an equilibrium described by

$$(1.2) \quad \Delta \varphi + g(x, \varphi) = \varphi u \quad x \in \Omega,$$

together with the Neumann boundary conditions (1.1).

In order to define an optimization problem for the steady state solution (1.2), we consider the cost

$$\int_\Omega c(x) u(x) \, dx.$$
Here $c(x)$ is the cost for a unit of fishing effort at the location $x \in \Omega$. Of course, the simplest choice here is $c(x) = \text{constant}$. However, one may have a cost $c(x)$ which increases with the distance of the point $x$ from the coastal city where fishing company has its base. In addition, if a region $\Omega_0 \subset \Omega$ is set aside as a marine park where no fishing is permitted, this can be modeled by setting $c(x) = +\infty$ for every $x \in \Omega_0$.

More generally, we consider a net profit which depends on the total fish caught, minus the harvesting cost:

$$J(\varphi, u) = \int_{\Omega} \varphi(x)u(x)dx - \Psi\left( \int_{\Omega} c(x) u(x) dx \right),$$

where $\Psi$ is a non decreasing, convex, and lower semicontinuous function such that $\Psi(0) = 0$, $\Psi'(0) = 1$.

The function $u = u(x)$ describes the harvesting strategy. It is reasonable to assume that it satisfies constraints of the form

$$u(x) \geq 0, \quad \int_{\Omega} b(x) u(x) dx \leq 1,$$

for some non-negative function $b(\cdot)$. The second constraint determines the maximum amount of harvesting power within the capabilities of the company. In practice, this may depend on the number of fishermen and on the size of fishing boats available.

Our main interest is in the existence and the qualitative properties of optimal solutions. This problem has two distinct features:

(i) In general, a given strategy $u(\cdot)$ does not determine a unique solution to the nonlinear elliptic problem (1.2), (1.1). In particular, one always has the trivial solution $\varphi \equiv 0$. For this reason, it is convenient to consider “optimal pairs” $(u, \varphi)$, where $u$ is an admissible harvesting strategy satisfying the constraints (1.5), while $\varphi$ is a positive solution of the corresponding elliptic problem (1.2), (1.1).

(ii) Since the cost functional (1.4) has only linear growth w.r.t. $u$, there is no guarantee that the optimal strategy $u(\cdot)$ will lie in the space $L^1(\Omega)$. Indeed, existence of optimal solutions will be proved within the larger space of bounded Radon measures supported on the closure $\overline{\Omega}$ of the domain. By deriving suitable necessary conditions for optimality, one can then understand whether the optimal measure $\mu$ is absolutely continuous w.r.t. Lebesgue measure.

The paper is organized as follows. In Section 2 we show the existence of a positive solution for the problem (1.2), (1.1). Section 3 is devoted to an averaging inequality satisfied by the positive solutions of (1.2), (1.1). In Section 4 we prove the upper semicontinuity of the cost functional $J$. Finally, in Section 5 the existence of an optimal measure-valued control is established.

Problems of optimal harvesting of a marine resource, governed by a semilinear elliptic equation, have been the subject of several investigations [2, 8, 10, 11, 13]. We remark that a quadratic harvesting cost such as

$$\int_{\Omega} c(x) u^2(x) dx$$

is entirely natural from a mathematical point of view, and guarantees that the optimal strategy is described by a function $u^{opt} \in L^2(\Omega)$. However, the linear cost (1.3) provides a more realistic model. In the one-dimensional case, the existence and a characterization of measure-valued optimal controls were proved in [6, 7]. The multi-dimensional case, studied in the present paper, requires a more careful analysis, relying on Sobolev space theory. For the basic theory of elliptic equations with measure-valued right hand side we refer to [3, 4, 9].
2. AN ELLIPTIC PROBLEM WITH MEASURE-VALUED COEFFICIENTS

Consider the semilinear elliptic problem
\begin{equation}
\begin{cases}
-\Delta \varphi + \varphi \mu = g(x, \varphi) & x \in \Omega, \\
\nabla \varphi \cdot n = 0 & x \in \partial \Omega.
\end{cases}
\end{equation}

We assume that \( \Omega \subset \mathbb{R}^N \) is a bounded connected open domain with smooth boundary. Moreover, \( g(x, \varphi) = f(x, \varphi) \varphi \), where \( f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) is a smooth function satisfying
\begin{equation}
\partial \varphi f(x, \varphi) < 0, \quad \text{for all } (x, \varphi) \in \overline{\Omega} \times \mathbb{R},
\end{equation}
\begin{equation}
f(x, \varphi) \geq 0, \quad \text{iff } \varphi < h(x),
\end{equation}
and \( h : \overline{\Omega} \to \mathbb{R} \) is a smooth nonnegative function. Since in (2.1) we allow \( \mu \) to be any nonnegative, bounded Radon measure on the closure \( \overline{\Omega} \), a solution to the elliptic boundary value problem (2.1) need not be smooth, or even continuous. We thus consider solutions of (2.1) in distributional sense. In the following, we denote by \( \frac{1}{\text{meas}(A)} \int_A f \, dx \) the average value of \( f \) on the set \( A \).

**Definition 2.1.** Let \( \varphi : \overline{\Omega} \to \mathbb{R} \) be an upper semicontinuous function whose pointwise values are determined by
\[ \varphi(x) = \lim_{r \downarrow 0} \int_{B(x, r) \cap \Omega} \varphi(y) \, dy. \]

We say \( \varphi \) is a solution of (2.1) if \( \varphi \in L^\infty(\Omega) \cap H^1(\Omega) \), and for every test function \( \phi \in C^2(\mathbb{R}^N) \) one has
\begin{equation}
\int_\Omega \nabla \phi \cdot \nabla \varphi \, dx + \int_\Omega \phi \varphi \, d\mu = \int_\Omega \phi g(x, \varphi) \, dx.
\end{equation}

**Remark 2.1.** Calling \( \kappa \doteq \sup_{x \in \overline{\Omega}, s \geq 0} g(x, s) \), the first equation in (2.1) implies that \( \varphi + \frac{\kappa}{2N} |x|^2 \) is subharmonic on \( \Omega \). Hence, up to a modification on a set of measure zero, \( \varphi \) is upper semicontinuous (see [1]). Multiplying the first equation in (2.1) by a test function \( \phi \) and integrating by parts, in view of the Neumann boundary conditions one obtains (2.4).

The main result of this section provides the existence and uniqueness of a nontrivial solution to the measure-valued elliptic problem (2.1).

**Theorem 2.1.** In the above setting, assume that
\begin{equation}
\mu \text{ is a bounded nonnegative Radon measure on } \overline{\Omega},
\end{equation}
\begin{equation}
\mu(A) = 0, \quad \text{for every } A \subset \overline{\Omega} \text{ with zero capacity},
\end{equation}
\begin{equation}
\mu(\overline{\Omega}) < \int_\Omega f(x, 0) \, dx.
\end{equation}

Then the boundary value problem (2.1) has a unique positive bounded solution such that
\begin{equation}
0 < \varphi(x) \leq M \quad \text{for a.e. } x \in \Omega,
\end{equation}
where
\[ M \doteq \sup_{x \in \overline{\Omega}} h(x), \]
Moreover, one has

\[
\int_{\Omega} \varphi \, d\mu = \int_{\Omega} g(x, \varphi) \, dx \leq M m_N(\Omega),
\]

(2.7)

\[
\int_{\Omega} |\nabla \varphi|^2 \, dx + \int_{\Omega} \varphi^2 \, d\mu = \int_{\Omega} \varphi g(x, \varphi) \, dx \leq M^2 m_N(\Omega),
\]

(2.8)

where \(m_N\) denotes the \(N\)-dimensional Lebesgue measure.

Toward a proof, we begin by regularizing the equation in (2.1) and studying the regularized problem. Since \(\mu\) is a nonnegative Radon measure on \(\Omega\), we can extend \(\mu\) to a nonnegative Radon measure on \(\mathbb{R}^N\) simply by setting \(\mu(\mathbb{R}^N \setminus \Omega) = 0\). Since \(\mu\) is zero on the sets with zero capacity there exist \(\tilde{\mu}, \mu\) such that

\[
\tilde{\mu}, \mu \geq 0, \quad \tilde{\mu} \in L^1_{\text{loc}}(\mathbb{R}^N), \quad \mu \in H^{-1}_{\text{loc}}(\mathbb{R}^N), \quad \mu = \tilde{\mu} + \mu.
\]

Consider two sequences of smooth functions with compact support \(\{\tilde{\mu}_n\}_{n \in \mathbb{N}}, \{\mu_n\}_{n \in \mathbb{N}} \subset C^\infty_c(\Omega)\) such that

\[
0 \leq \tilde{\mu}_n \leq \tilde{\mu}, \quad \mu_n \geq 0, \quad \tilde{\mu}_n \to \tilde{\mu} \text{ in } L^1_{\text{loc}}(\mathbb{R}^N), \quad \mu_n \to \mu \text{ in } H^{-1}_{\text{loc}}(\mathbb{R}^N).
\]

Here and in the sequel, we use the notation

\[
\mu_n = \tilde{\mu}_n + \mu_n.
\]

Since \(\|\mu_n\|_{L^1(\Omega)} = \mu_n(\Omega) \to \mu(\Omega)\), in view of (2.5) we can also assume that

\[
\mu_n(\Omega) < \int_{\Omega} f(x, 0) \, dx \quad \text{for all } n \in \mathbb{N}.
\]

We now study the approximated problem with smooth coefficients

\[
\begin{cases}
-\Delta \varphi_n + \mu_n \varphi_n = g(x, \varphi_n), & \text{in } \Omega, \\
\partial_n \varphi_n = 0, & \text{on } \partial \Omega,
\end{cases}
\]

(2.13)

where \(\partial_n\) denotes the derivative in the direction of the outer normal to the boundary \(\partial \Omega\).

\textbf{Lemma 2.1.} The boundary value problem (2.13) has a positive smooth bounded solution \(\varphi_n = \varphi_n(x)\) such that

\[
0 < \delta_n \leq \varphi_n(x) \leq M,
\]

(2.14)

where \(\delta_n\) is a positive constant, possibly depending on \(n\).

\textbf{Proof.} Since \(g(\cdot, M) \leq 0\), the constant function always equal to \(M\) is a supersolution of (2.13).

In order to construct a subsolution, we introduce the functional

\[
I_n(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} \left[\mu_n(x) - f(x, 0)\right] u^2 \, dx, \quad u \in H^1(\Omega),
\]

and the manifold of codimension 1

\[
\mathcal{B} = \left\{ u \in H^1(\Omega); \int_{\Omega} u^2 \, dx = 1 \right\}.
\]

Since \(\mu_n\) and \(f(x, \cdot)\) are smooth, the following facts are clear:

(i) \(I_n\) is a continuous, quadratic functional on \(H^1(\Omega)\).

(ii) \(I_n\) is coercive on \(\mathcal{B}\). Namely, if \(\{u_k\}_k \subset \mathcal{B}\) and \(\lim_{k \to \infty} \|u_k\|_{H^1(\Omega)} = \infty\), then \(\lim_{k \to \infty} I_n(u_k) = \infty\).
(iii) $I_n$ is weakly lower semicontinuous in $H^1(\Omega)$. Namely
\[ u_k \rightharpoonup u \text{ weakly in } H^1(\Omega) \implies I_n(u) \leq \liminf_{k \to \infty} I_n(u_k). \]

Therefore, restricted to $\mathcal{B}$, the functional $I_n$ is bounded below. Since $\mathcal{B} \subset H^1(\Omega)$ is weakly closed, by taking the limit of a minimizing sequence we obtain a function $u_n \in \mathcal{B}$ such that
\[ I_n \doteq \inf_{u \in \mathcal{B}} I_n(u) = I_n(u_n). \]

The necessary conditions for optimality yield the existence of a scalar Lagrange multiplier $\eta_n$ such that
\[ \text{the Gateaux derivative of } I_n \text{ at } u_n \text{ satisfies } \]
\[ DI_n(u_n) - \eta_n u_n = 0. \]

Observing that $I_n$ is positively homogeneous of degree 2, we have
\[ \eta_n = \frac{d}{d\lambda} I_n(\lambda u_n) \bigg|_{\lambda=1} = \frac{d}{d\lambda} \left[ \lambda I_n(u_n) \right] \bigg|_{\lambda=1} = 2I_n(u_n). \]

This means that $u_n$ and $\eta_n$ solve the linear eigenvalue problem
\[ \begin{cases} -\Delta u_n + \mu_n u_n = u_n f(x,0) + \eta_n u_n, & \text{in } \Omega, \\ \partial_n u_n = 0, & \text{on } \partial \Omega, \end{cases} \]
where
\[ \eta_n = 2I(u_n). \]

Since the map with constant value $\frac{1}{\sqrt{m_N(\Omega)}}$ belongs to $\mathcal{B}$ we have (see (2.12))
\[ \eta_n = 2 \min_{u \in \mathcal{B}} I_n(u) \leq 2I_n \left( \frac{1}{\sqrt{m_N(\Omega)}} \right) = \frac{1}{m_N(\Omega)} \left( \int_\Omega \mu_n dx - \int_\Omega f(x,0) dx \right) < 0. \]

Classical results on linear elliptic eigenvalue problems [12] yield
\[ u_n \in L^\infty(\Omega), \quad u_n \geq c_n > 0, \]
for some positive constant $c_n$. As subsolution for the problem (2.13) we choose the function $k_n u_n$, where $k_n$ is a positive constant that will be chosen later. Imposing
\[ k_n \leq \frac{M}{\|u_n\|_{L^\infty(\Omega)}}, \quad n \in \mathbb{N}, \]
we have
\[ 0 < k_n u_n \leq M, \quad n \in \mathbb{N}. \]

Observe that
\[ -\Delta(k_n u_n) + \mu_n k_n u_n - g(x,k_n u_n) = \]
\[ = k_n \left( -\Delta u_n + \mu_n u_n \right) - k_n u_n f(x,k_n u_n) \]
\[ = k_n \left( u_n f(x,0) + \eta_n u_n \right) - k_n u_n f(x,k_n u_n) \]
\[ = k_n u_n \left( f(x,0) - f(x,k_n u_n) \right) + \eta_n. \]

Due to the boundedness of $u_n$, the monotonicity of $f(x,\cdot)$, and (2.15), we can choose $k_n$ so small that
\[ (f(x,0) - f(x,k_n u_n)) + \eta_n \leq \frac{\eta_n}{2} < 0. \]

Therefore $k_n u_n$ is a subsolution of (2.13). Due to (2.16) there exists a solution $\varphi_n$ of (2.13) such that
\[ 0 < c_n k_n \leq k_n u_n \leq \varphi_n \leq M. \]
Proof of Theorem 2.1. 1. Let \( \{ \varphi_n \}_{n \in \mathbb{N}} \) be a sequence of maps constructed in Lemma 2.1. Due to (2.14), the functions \( \varphi_n \) are uniformly bounded. We claim that
\[
\int_{\Omega} |\nabla \varphi_n|^2 \, dx \leq M \| g \|_{L^\infty(\Omega \times (0,M))} \, m(\Omega), \quad n \in \mathbb{N}.
\]
Indeed, multiplying (2.13) by \( \varphi_n \) and integrating on \( \Omega \) one obtains
\[
0 = -\int_{\Omega} \varphi_n \Delta \varphi_n \, dx + \int_{\Omega} \varphi_n^2 \mu_n \, dx - \int_{\Omega} \varphi_n g(x, \varphi_n) \, dx \geq 0
\]
\[
\geq -\int_{\partial \Omega} \varphi_n \partial_n \varphi_n \, dS + \int_{\Omega} |\nabla \varphi_n|^2 \, dx - \int_{\Omega} \varphi_n g(x, \varphi_n) \, dx = \int_{\Omega} |\nabla \varphi_n|^2 \, dx - \int_{\Omega} \varphi_n g(x, \varphi_n) \, dx.
\]
Here \( dS \) is the \( N - 1 \) dimensional measure on \( \partial \Omega \). Therefore
\[
\int_{\Omega} |\nabla \varphi_n|^2 \, dx \leq \int_{\Omega} \varphi_n g(x, \varphi_n) \, dx,
\]
and (2.17) follows from (2.14).

2. Since the sequence \( \{ \varphi_n \}_{n \in \mathbb{N}} \) is bounded in \( L^\infty(\Omega) \) and also in \( H^1(\Omega) \), by possibly taking a subsequence we can find \( \varphi \in L^\infty(\Omega) \cap H^1(\Omega) \) such that
\[
\varphi_n \rightharpoonup \varphi \quad \text{weakly in } H^1(\Omega),
\]
\[
\varphi_n \rightarrow \varphi \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega.
\]
In particular,
\[
\nabla \varphi_n \rightharpoonup \nabla \varphi \quad \text{weakly in } L^2(\Omega).
\]
To prove that \( \varphi \) solves (2.1), let \( \phi \in C^\infty(\mathbb{R}^N) \) be a test function. From (2.13) it follows
\[
\int_{\Omega} \nabla \varphi_n \cdot \nabla \phi \, dx + \int_{\Omega} \phi \varphi_n \mu_n \, dx = \int_{\Omega} g(x, \varphi_n) \phi \, dx.
\]
Letting \( n \to \infty \), thanks to (2.9), (2.10), (2.11), (2.18), (2.19), and the Dominated Convergence Theorem, one has
\[
\int_{\Omega} \nabla \varphi \cdot \nabla \phi \, dx + \int_{\Omega} \phi \varphi d\mu = \int_{\Omega} g(x, \varphi) \phi \, dx,
\]
showing that \( \varphi \) is a solution of (2.1).

3. Clearly, from (2.14) it follows \( 0 \leq \varphi \leq M \). We now prove that \( \varphi \) is strictly positive. As a first step we show that
\[
\varphi \not\equiv 0.
\]
Assume by contradiction that \( \varphi(x) = 0 \) for all \( x \in \Omega \). Then
\[
\varphi_n \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega).
\]
By (2.13) we have
\[
-\frac{\Delta \varphi_n}{\varphi_n} = f(x, \varphi_n) - \mu_n.
\]
Integrating over \( \Omega \) and using the Neumann boundary conditions one obtains
\[
\int_{\Omega} \frac{|\nabla \varphi_n|^2}{\varphi_n^2} \, dx = \int_{\Omega} \mu_n \, dx - \int_{\Omega} f(x, \varphi_n) \, dx.
\]
Letting $n \to \infty$, thanks to (2.5), (2.14), and (2.18), one obtains
\[
\lim_{n \to \infty} \int_{\Omega} \frac{|\nabla \varphi_n|^2}{\varphi_n^2} dx = \mu(\Omega) - \int_{\Omega} f(x,0) dx < 0,
\]
reaching to a contradiction. Therefore (2.20) is proved.

To prove that $\varphi$ is strictly positive, we observe that
\[
-\frac{\Delta \varphi_n}{\varphi_n} = -\text{div} \left( \frac{\nabla \varphi_n}{\varphi_n} \right) - \frac{|\nabla \varphi_n|^2}{\varphi_n^2} = -\Delta(\log(\varphi_n)) - \frac{|\nabla \varphi_n|^2}{\varphi_n^2}.
\]
yields
\[
-\Delta(\log(\varphi_n)) = f(x, \varphi_n) - \mu_n + \frac{|\nabla \varphi_n|^2}{\varphi_n^2}.
\]
Thanks to (2.10), (2.14), and (2.21) the right hand side is bounded in $L^1(\Omega)$, hence
\[
\{\Delta(\log(\varphi_n))\}_{n \in \mathbb{N}} \text{ is bounded in } L^1(\Omega).
\]
By the analysis in [14], this implies
\[
\{\log(\varphi_n)\}_{n \in \mathbb{N}} \text{ is bounded in } W^{1,q}(\Omega), 1 \leq q < \frac{N}{N-1}.
\]
Therefore there exists a function $\psi \in W^{1,q}(\Omega), 1 \leq q < \frac{N}{N-1}$ such that, passing to a subsequence,
\[
\log(\varphi_n) \to \psi \quad \text{weakly in } W^{1,q}(\Omega), 1 \leq q < \frac{N}{N-1},
\]
\[
\log(\varphi_n) \to \psi \quad \text{strongly in } L^q(\Omega), 1 \leq q < \frac{N}{N-1} \text{ and a.e. in } \Omega.
\]
We now have
\[
\psi(x) > -\infty \quad \text{and} \quad \varphi(x) = e^{\psi(x)} > 0 \text{ for a.e. } x \in \Omega
\]
proving our claim.

\textbf{4.} We now prove the uniqueness of the positive solutions of (2.1) satisfying (2.6). The main idea is taken from [5]. Assume that there exist two positive solutions $\varphi$ and $\overline{\varphi}$ of (2.1), so that
\[
-\frac{\Delta \varphi}{\varphi} + \mu = f(x, \varphi), \quad -\frac{\Delta \overline{\varphi}}{\varphi} + \mu = f(x, \overline{\varphi}),
\]
and hence
\[
-\frac{\Delta \varphi}{\varphi} + \frac{\Delta \overline{\varphi}}{\varphi} = f(x, \varphi) - f(x, \overline{\varphi}).
\]
Multiplying by $\varphi^2 - \overline{\varphi}^2$ and integrating over $\Omega$, using (2.3), (2.2), (2.6), and the Neumann boundary conditions one obtains
\[
0 = \int_{\Omega} \left( -\frac{\Delta \varphi}{\varphi} + \frac{\Delta \overline{\varphi}}{\varphi} \right) (\varphi^2 - \overline{\varphi}^2) dx - \int_{\Omega} (f(x, \varphi) - f(x, \overline{\varphi})) (\varphi^2 - \overline{\varphi}^2) dx
\]
\[
= \int_{\Omega} \left( -\varphi \Delta \varphi + \frac{\varphi^2}{\varphi} \Delta \varphi + \frac{\varphi^2}{\varphi} \Delta \overline{\varphi} - \overline{\varphi} \Delta \overline{\varphi} \right) dx
\]
\[
- \int_{\Omega} (f(x, \varphi) - f(x, \overline{\varphi})) (\varphi - \overline{\varphi}) (\varphi + \overline{\varphi}) dx
\]
\[
= \int_{\Omega} \left( |\nabla \varphi|^2 - \frac{2 \varphi}{\varphi} \nabla \varphi \cdot \nabla \varphi + \frac{\varphi^2}{\varphi^2} |\nabla \varphi|^2 - \frac{2 \varphi}{\varphi} \nabla \overline{\varphi} \cdot \nabla \overline{\varphi} + \frac{\varphi^2}{\varphi^2} |\nabla \overline{\varphi}|^2 + |\nabla \overline{\varphi}|^2 \right) dx
\]
\[
+ \int_{\partial \Omega} \left( -\varphi \partial_n \varphi - \frac{\varphi^2}{\varphi} \partial_n \varphi + \frac{\varphi}{\varphi} \partial_n \overline{\varphi} - \overline{\varphi} \partial_n \overline{\varphi} \right) dS
\]
\[
= 0
\]
\[ -\int_{\Omega} \int_{0}^{1} \frac{\partial f (x, \theta \varphi + (1 - \theta) \bar{\varphi}) (\varphi - \bar{\varphi})^2 (\varphi + \bar{\varphi})}{\theta < 0} d\theta dx > 0 \text{ a.e.} \]

\[ \geq \int_{\Omega} \left( \left| \nabla \varphi \right|^2 - \frac{2\varphi}{\varphi} \nabla \varphi \cdot \nabla \bar{\varphi} + \frac{\varphi^2}{\varphi^2} |\nabla \varphi|^2 \right) + \left( \left| \nabla \bar{\varphi} \right|^2 - \frac{2\bar{\varphi}}{\varphi} \nabla \varphi \cdot \nabla \bar{\varphi} + \frac{\bar{\varphi}^2}{\varphi^2} |\nabla \varphi|^2 \right) \right) dx \]

\[ + \inf_{\Omega \times [0, M]} |\partial f| \int_{\Omega} (\varphi - \bar{\varphi})^2 (\varphi + \bar{\varphi}) dx \]

\[ = \int_{\Omega} \left( \left| \nabla \varphi - \frac{\varphi}{\varphi} \nabla \bar{\varphi} \right|^2 + \left| \nabla \bar{\varphi} - \frac{\varphi}{\varphi} \nabla \varphi \right|^2 \right) dx + \inf_{\Omega \times [0, M]} |\partial f| \int_{\Omega} (\varphi - \bar{\varphi})^2 (\varphi + \bar{\varphi}) dx \]

\[ \geq \inf_{\Omega \times [0, M]} |\partial f| \int_{\Omega} (\varphi - \bar{\varphi})^2 (\varphi + \bar{\varphi}) dx \geq 0 \]

Therefore \( \varphi = \bar{\varphi} \) for a.e. \( x \in \Omega \), proving uniqueness of positive solutions.

**5.** We conclude proving the estimates (2.7) and (2.8). Let \( \varphi \in L^\infty(\Omega) \cap H^1(\Omega) \) be the positive bounded solution of (2.1).

Using the function \( \phi \equiv 1 \) as test function in (2.4) we have

\[ \int_{\Omega} \varphi d\mu = \int_{\Omega} g(x, \varphi) dx, \]

therefore (2.7) follows from (2.6).

Due to the density of \( C^2(\bar{\Omega}) \) in \( L^\infty(\Omega) \cap H^1(\Omega) \), we can find a sequence of test functions \( \{\phi_\nu\}_{\nu \in \mathbb{N}} \subset C^2(\mathbb{R}^N) \) such that

\[ \phi_\nu \longrightarrow \varphi \quad \text{a.e. in } \Omega \text{ and in } H^1(\Omega) \text{ as } \nu \to \infty. \]

By (2.4) one has

\[ \int_{\Omega} \nabla \phi_\nu \cdot \nabla \varphi dx + \int_{\Omega} \phi_\nu \varphi d\mu = \int_{\Omega} \phi_\nu g(x, \varphi) dx. \]

letting \( \nu \to \infty \) we get

\[ \int_{\Omega} |\nabla \varphi|^2 dx + \int_{\Omega} \varphi^2 d\mu = \int_{\Omega} \varphi g(x, \varphi) dx. \]

Therefore (2.8) follows from (2.6).

\[ \square \]

**3. An Averaging Formula**

Let \( \{\mu_n\}_{n \in \mathbb{N}} \) be a bounded sequence of nonnegative finite measures on \( \bar{\Omega} \) satisfying (2.5). For any \( n \in \mathbb{N} \), let \( \varphi_n \) be the unique positive solution of the Neumann problem (see Theorem 2.1)

(3.1)
\[
\begin{aligned}
-\Delta \varphi_n + \mu_n \varphi_n &= g(x, \varphi_n) & \text{in } \Omega, \\
\partial_n \varphi_n &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

Due to the boundedness of \( \{\mu_n\}_{n \in \mathbb{N}} \) there exists a nonnegative bounded Radon measure \( \mu \) on \( \bar{\Omega} \) such that, passing to a subsequence,

(3.2)

\[ \mu_n \rightharpoonup \mu \quad \text{in the sense of measures on } \bar{\Omega}, \]

namely

(3.3)

\[ \int_{\bar{\Omega}} \phi d\mu_n \rightharpoonup \int_{\bar{\Omega}} \phi d\mu, \quad \phi \in C(\bar{\Omega}). \]
Moreover, thanks to Theorem 2.1, there exists a nonnegative function \( \varphi \in L^\infty(\Omega) \cap H^1(\Omega) \) such that, passing to a subsequence,

\[
\varphi_n \rightharpoonup \varphi \quad \text{strongly in } L^p(\Omega), \quad 1 \leq p < \infty, \quad \text{and a.e. in } \Omega,
\]

\[
\varphi_n \to \varphi \quad \text{weakly in } H^1(\Omega).
\]

We shall need the following version of the mean value inequality, valid for solution of (3.1). In the following, \( B_r(x) \) denotes the open ball centered at \( x \) with radius \( r \). The Lebesgue measure of the \( N \)-dimensional unit ball is denoted by \( \omega_N = m_N(B_1(0)) \). Of course, this implies \( m_N(B_r(x)) = \omega_N r^N \).

**Lemma 3.1.** For every \( x \in \overline{\Omega} \) and \( r > 0 \), one has

\[
\varphi_n(x) \leq \frac{1}{\omega(x)\omega_N r^N} \int_{B_r(x) \cap \Omega} \varphi_n(y)dy - \frac{1}{\omega(x)\omega_N r^N} \int_0^r \int_0^\rho \frac{\rho^{N-2} m_{N-2}(\partial B_s(x) \cap \partial \Omega)}{s^{N-2}} \alpha(s, x) ds d\rho + \frac{\kappa}{\omega(x)\omega_N r^N} \int_0^r \int_0^\rho \frac{\rho^{N-1} m_N(B_s(x) \cap \Omega)}{s^{N-1}} ds d\rho,
\]

where

\[
\alpha(r, x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ \frac{1}{2} & \text{if } x \in \partial \Omega. \end{cases}
\]

Let us comment the three terms and the coefficients that appear in (3.5). The last term takes into account the fact that \( \varphi_n + \kappa \frac{r^2}{2N} |x|^2 \) (and not \( \varphi_n \)) is subharmonic (see Remark 2.1). Moreover, since \( \Omega \) is a bounded set, we have to intersect the ball \( B_r(x) \) with \( \overline{\Omega} \) and use the fact that \( \partial_n \varphi_n = 0 \) on \( \partial \Omega \). The coefficients take into account this fact. Finally, if the boundary \( \partial \Omega \) is not flat, the map

\[
r \mapsto m_{N-2}(\partial B_r(x) \cap \partial \Omega)
\]

varies according to the curvature of \( \partial \Omega \). This fact is accounted by the second term of the right hand side of (3.5). We also observe that, if \( x \in \Omega \) and \( B_r(x) \subset \Omega \), then

\[
\frac{m_N(B_r(x) \cap \Omega)}{\omega(x)r^N \omega_N} = 1, \quad \alpha(\cdot, x) = 0,
\]

\[
\frac{\kappa}{\omega(x)r^N \omega_N} \int_0^r \int_0^\rho \frac{\rho^{N-1} m_N(B_s(x) \cap \Omega)}{s^{N-1}} ds d\rho = \frac{\kappa \omega_N r^{N+2}}{2r^N (N+2) \omega_N} = \frac{\kappa r^2}{2(N+2)}.
\]

From (3.5) we thus recover the usual average inequality for subharmonic functions (recall that \( \varphi_n + \frac{\kappa}{2} |x|^2 \) and not \( \varphi_n \) is subharmonic):

\[
\varphi_n(x) \leq \int_{B_r(x)} \varphi_n(y)dy + \frac{\kappa r^2}{2(N+2)}.
\]

On the other hand, if \( x \in \partial \Omega \), then as \( r \to 0 \), the intersection \( \partial B_r(x) \cap \Omega \) tends to a half sphere, while \( n(y) \) and \( \kappa \frac{r^2}{2N} |x|^2 \) become nearly orthogonal. Therefore,

\[
\lim_{r \to 0} \frac{m_N(B_r(x) \cap \Omega)}{\omega(x)r^N \omega_N} = 1, \quad \lim_{r \to 0} \alpha(r, x) = 0,
\]
\[
\lim_{r \to 0} \frac{\kappa}{\omega(x) r^N \omega_N} \int_0^r \int_0^{\rho \cdot N - 1} \frac{m_N(B_\rho(x) \cap \Omega)}{s^{N-1}} ds d\rho = \lim_{r \to 0} \frac{\kappa r^2}{N + 2} = 0.
\]

**Proof.** Fix \( x \in \overline{\Omega} \) and \( r > 0 \). From Remark 2.1 and (3.1) it follows

\[
0 \leq \int_{B_r(x) \cap \Omega} \Delta \left( \varphi_n + \frac{\kappa}{2N} |y|^2 \right) dy = \int_{B_r(x) \cap \Omega} \Delta \varphi_n dy + \kappa m_N(B_r(x) \cap \Omega)
\]

\[
= \int_{B_r(x) \cap \Omega} \frac{\partial_n \varphi_n dm_{N-1}}{\partial B_r(x) \cap \Omega} + \int_{\partial B_r(x) \cap \Omega} \nabla \varphi_n(y) \cdot \frac{y - x}{r} dm_{N-1}(y) + \kappa m_N(B_r(x) \cap \Omega)
\]

\[
= \int_{\partial B_r(x)} \nabla \varphi_n(y) \cdot \frac{y - x}{r} \chi_{\Omega}(y) dm_{N-1}(y) + \kappa m_N(B_r(x) \cap \Omega)
\]

\[
= r^{N-1} \int_{\partial B_1(0)} \nabla \varphi_n(x + ry) \cdot y \chi_{\Omega}(x + ry) dm_{N-1}(y) + \kappa m_N(B_r(x) \cap \Omega)
\]

\[
= r^{N-1} \int_{\partial B_1(0)} \partial_r \varphi_n(x + ry) \chi_{\Omega}(x + ry) dm_{N-1}(y) + \kappa m_N(B_r(x) \cap \Omega)
\]

\[
= r^{N-1} \partial_r \left( r^{N-1} \int_{\partial B_1(0)} \varphi_n(y) dm_{N-1}(y) \right)
\]

\[
- r \int_{\partial B_r(x) \cap \Omega} \varphi_n(y) \mathbf{n}(y) \cdot \frac{y - x}{r} dm_{N-2}(y) + \kappa m_N(B_r(x) \cap \Omega)
\]

\[
= r^{N-1} \partial_r \left( \frac{m_{N-1}(\partial B_r(x) \cap \Omega)}{r^{N-1}} \int_{\partial B_r(x) \cap \Omega} \varphi_n(y) dm_{N-1}(y) \right)
\]

\[
- r m_{N-2}(\partial B_r(x) \cap \partial \Omega) \int_{\partial B_r(x) \cap \Omega} \varphi_n(y) \mathbf{n}(y) \cdot \frac{y - x}{r} dm_{N-2}(y) + \kappa m_N(B_r(x) \cap \Omega).
\]

Therefore,

\[
0 \leq \partial_r \left( \frac{m_{N-1}(\partial B_r(x) \cap \Omega)}{r^{N-1}} \int_{\partial B_r(x) \cap \Omega} \varphi_n(y) dm_{N-1}(y) \right)
\]

\[
- \frac{m_{N-2}(\partial B_r(x) \cap \partial \Omega)}{r^{N-2}} \alpha(r, x) + \kappa \frac{m_N(B_r(x) \cap \Omega)}{r^{N-1}}.
\]

Since, if \( x \in \partial \Omega \), as \( r \to 0 \), \( \partial B_r(x) \cap \Omega \) tends to a half sphere, we have

\[
\lim_{r \to 0} \int_{\partial B_r(x) \cap \Omega} \varphi_n(y) dm_{N-1}(y) = \varphi_n(x),
\]

\[
\lim_{r \to 0} \frac{m_{N-1}(\partial B_r(x) \cap \Omega)}{r^{N-1}} = \frac{1}{2} m_{N-1}(\partial B_1(0)).
\]

Hence, integrating (3.6) w.r.t. \( r \) we get

\[
\omega(x) m_{N-1}(\partial B_1(0)) \varphi_n(x) \leq \frac{m_{N-1}(\partial B_r(x) \cap \Omega)}{r^{N-1}} \int_{\partial B_r(x) \cap \Omega} \varphi_n(y) dm_{N-1}(y)
\]

\[
- \int_0^r \frac{m_{N-2}(\partial B_{\rho}(x) \cap \partial \Omega)}{\rho^{N-2}} \alpha(\rho, x) d\rho
\]

\[
+ \kappa \int_0^r \frac{m_N(B_{\rho}(x) \cap \Omega)}{\rho^{N-1}} d\rho,
\]
that is

$$\varphi_n(x) \leq \frac{m_{N-1}((\partial B_r(x) \cap \Omega))}{r^{N-1} \omega(x) m_{N-1}(\partial B_1(0))} \int_{\partial B_r(x) \cap \Omega} \varphi_n(y)dm_{N-1}(y)$$

$$- \frac{1}{\omega(x)m_{N-1}(\partial B_1(0))} \int_0^r \frac{m_{N-2}(\partial B_\rho(x) \cap \partial \Omega)}{\rho^{N-2}} \alpha(\rho, x) d\rho$$

$$+ \frac{\kappa}{\omega(x)m_{N-1}(\partial B_1(0))} \int_0^r \frac{m_N(B_\rho(x) \cap \Omega)}{\rho^{N-1}} d\rho.$$

(3.7)

We continue by evaluating the average of $\varphi_n$ on $B_r(x) \cap \Omega$. In light of (3.7)

$$\int_{B_r(x) \cap \Omega} \varphi_n(y) dy = \int_0^r \int_{\partial B_\rho(x) \cap \Omega} \varphi_n(y)dm_{N-1}(y) d\rho$$

$$\geq \int_0^r \varphi_n(x) \rho^{N-1} \omega(x)m_{N-1}(\partial B_1(0)) d\rho$$

$$+ \int_0^r \int_0^\rho \rho^{N-1} \frac{m_{N-2}(\partial B_s(x) \cap \partial \Omega)}{s^{N-2}} \alpha(s, x) ds d\rho$$

$$- \kappa \int_0^r \int_0^\rho \rho^{N-1} \frac{m_N(B_s(x) \cap \Omega)}{s^{N-1}} ds d\rho$$

$$= \varphi_n(x) \frac{r^N}{N} \omega(x)m_{N-1}(\partial B_1(0))$$

$$+ \int_0^r \int_0^\rho \rho^{N-1} \frac{m_{N-2}(\partial B_s(x) \cap \partial \Omega)}{s^{N-2}} \alpha(s, x) ds d\rho$$

$$- \kappa \int_0^r \int_0^\rho \rho^{N-1} \frac{m_N(B_s(x) \cap \Omega)}{s^{N-1}} ds d\rho.$$

Using the identity

$$m_N(B_1(0)) = \frac{m_{N-1}(\partial B_1(0))}{N},$$

we have

$$\varphi_n(x) \leq \frac{m_N(B_1(x) \cap \Omega)}{\omega(x) r N \omega_N} \int_{B_r(x) \cap \Omega} \varphi_n(y) dy$$

$$- \frac{1}{\omega(x) r N \omega_N} \int_0^r \int_0^\rho \rho^{N-1} \frac{m_{N-2}(\partial B_s(x) \cap \partial \Omega)}{s^{N-2}} \alpha(s, x) ds d\rho$$

$$+ \frac{\kappa}{\omega(x) r N \omega_N} \int_0^r \int_0^\rho \rho^{N-1} \frac{m_N(B_s(x) \cap \Omega)}{s^{N-1}} ds d\rho,$$

that is (3.5).

\[\square\]

4. Semicontinuity of the Payoff Functional

The following upper semicontinuity result provides the key ingredient in the proof of existence of a global maximizer for the payoff functional $J$ in (1.4).

We consider again a bounded sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of nonnegative finite measures on $\overline{\Omega}$ satisfying (2.5). For any $n \in \mathbb{N}$, let $\varphi_n$ be the unique positive solution of the Neumann problem (3.1). Due to the boundedness of $\{\mu_n\}_{n \in \mathbb{N}}$ there exists a nonnegative bounded Radon measure $\mu$ on $\overline{\Omega}$ such that, passing to a subsequence, (3.2) holds. Moreover, thanks to Theorem 2.1, there exists a nonnegative function $\varphi \in L^\infty(\Omega) \cap H^1(\Omega)$ such that, passing to a subsequence, (3.4) holds.
Lemma 4.1. In the above setting, one has
\[
\limsup_{n \to \infty} \int_{\Omega} \varphi_n(x) d\mu_n \leq \int_{\Omega} \varphi(x) d\mu.
\]

Proof. Let \( \phi \in C(\overline{\Omega}) \), \( \phi \geq 0 \). Since
\[
\int_{\Omega} \phi \varphi d\mu = \inf_{\psi \in C(\overline{\Omega}), \psi \geq \varphi} \int_{\Omega} \psi \varphi d\mu,
\]
we fix \( \psi \in C(\overline{\Omega}) \) such that \( \psi \geq \varphi \) and prove that
\[
(4.1) \quad \limsup_{n} \int_{\Omega} \phi \varphi_n(x) d\mu_n \leq \int_{\Omega} \phi \psi d\mu.
\]
Due to (3.5), the boundedness of \( \{ \mu_n \}_{n \in \mathbb{N}} \), and the Hölder inequality, we have
\[
\int_{\Omega} \phi(x) \varphi_n(x) d\mu_n(x)
\]
\[
\leq \frac{m_N(B_r(x) \cap \Omega) \phi(x)}{\omega(x)^r N \omega_N} \int_{B_r(x) \cap \Omega} \varphi_n(y) dy d\mu_n(x)
\]
\[
- \int_{\Omega} \frac{\phi(x)}{\omega(x)^r N \omega_N} \int_0^r \int_0^\rho^N \frac{\alpha(s, x)}{s^{N-2}} dsd\rho d\mu_n(x)
\]
\[
+ \int_{\Omega} \frac{\kappa \phi(x)}{\omega(x)^r N \omega_N} \int_0^r \int_0^\rho^N \frac{\alpha(s, x)}{s^{N-1}} dsd\rho d\mu_n(x)
\]
\[
= \int_{\Omega} \frac{m_N(B_r(x) \cap \Omega) \phi(x)}{\omega(x)^r N \omega_N} \int_{B_r(x) \cap \Omega} (\varphi_n(y) - \varphi(y)) dy d\mu_n(x)
\]
\[
+ \int_{\Omega} \frac{m_N(B_r(x) \cap \Omega) \phi(x)}{\omega(x)^r N \omega_N} \int_{B_r(x) \cap \Omega} \varphi(y) dy d\mu_n(x)
\]
\[
- \int_{\Omega} \frac{\phi(x)}{\omega(x)^r N \omega_N} \int_0^r \int_0^\rho^N \frac{m_{N-2}(\partial B_s(x) \cap \partial \Omega)}{s^{N-2}} dsd\rho d\mu_n(x)
\]
\[
+ \int_{\Omega} \frac{\kappa \phi(x)}{\omega(x)^r N \omega_N} \int_0^r \int_0^\rho^N \frac{m_{N-1}(B_s(x) \cap \Omega)}{s^{N-1}} dsd\rho d\mu_n(x)
\]
\[
\leq c \left\| \varphi_n - \varphi \right\|_{L^r(\Omega)} + \frac{m_N(B_r(x) \cap \Omega) \phi(x)}{\omega(x)^r N \omega_N} \int_{B_r(x) \cap \Omega} \psi(y) dy d\mu_n(x) + cr + cr^2,
\]
for some constant \( c \geq 0 \) independent on \( n \) and \( r \). As \( n \to \infty \) thanks to the continuity of \( \psi \), (3.3), and (3.4),
\[
(4.2) \quad \limsup_{n} \int_{\Omega} \phi(x) \varphi_n(x) d\mu_n(x) \leq \int_{\Omega} m_N(B_r(x) \cap \Omega) \phi(x) \frac{\psi(y) dy d\mu(x) + cr + cr^2}{\omega(x)^r N \omega_N}
\]
Since
\[
\lim_{r \to 0} \frac{m_N(B_r(x) \cap \Omega)}{\omega(x)^r N \omega_N} = 1, \quad x \in \overline{\Omega},
\]
sending $r \to 0$ in (4.2), using the continuity of $\psi$, we get
\[
\limsup_n \int_\Omega \phi(x) \varphi_n(x) d\mu_n(x) \leq \int_\Omega \phi(x) \psi(x) d\mu(x),
\]
that is (4.1). \hfill \square

5. Existence of an optimal measure-valued control

We discuss here the existence of an optimal pair $(\varphi^*, \mu^*)$ for the problem
\[
\text{maximize: } J(\varphi, \mu) = \int_\Omega \varphi(x) d\mu(x) - \Psi \left( \int_\Omega c(x) d\mu(x) \right),
\]
where $\varphi$ is the unique strictly positive solution of
\[
\begin{cases}
\Delta \varphi(x) + g(x, \varphi) = \varphi \cdot \mu, & x \in \Omega, \\
\nabla \varphi \cdot \mathbf{n} = 0, & x \in \partial \Omega,
\end{cases}
\]
and $\mu$ is a non-negative Radon measure on $\overline{\Omega}$ which satisfies
\[
\int_\Omega b d\mu \leq 1.
\]
We assume that the functions $b : \overline{\Omega} \to \mathbb{R}$ and $c : \overline{\Omega} \to \mathbb{R}$ are both lower semi-continuous, and satisfy
\[
b(x) \geq 0, \quad c(x) \geq c_0 > 0, \quad \text{for all } x \in \overline{\Omega},
\]
for some positive constant $c_0$.

**Theorem 5.1.** In the above setting, the constrained maximization problem (5.1)–(5.2) admits an optimal solution.

**Proof.** Let $\{(\varphi_n, \mu_n)\}_{n \in \mathbb{N}}$ be a maximizing sequence, where $\mu_n$ satisfies (2.5) and $\varphi_n$ is the corresponding positive solution of (2.1).

It is not restrictive to assume that
\[
\text{supp}(\mu_n) \subset A_n = \{ x \in \overline{\Omega}; \varphi_n(x) \geq c_0 \}.
\]
Otherwise we could consider the measure
\[
\mu'_n = \mu_n \cdot \chi_{A_n}.
\]
Clearly, $\mu'_n$ satisfies (5.2) and the function $\varphi_n$ provides a subsolution to the equation (2.1), with $\mu_n$ replaced by $\mu'_n$. Therefore, we can find a solution $\varphi'_n \geq \varphi_n$ of the same problem. We thus have
\[
J(\varphi_n, \mu_n) - J(\varphi'_n, \mu'_n) = \int_\Omega \varphi_n d\mu_n - \int_\Omega \varphi'_n d\mu'_n - \Psi \left( \int_\Omega c d\mu_n \right) + \Psi \left( \int_\Omega c d\mu'_n \right) \\
= \int_\Omega \varphi_n d\mu_n - \int_{\Omega \setminus A_n} \varphi'_n d\mu_n - \Psi \left( \int_\Omega c d\mu_n \right) + \Psi \left( \int_\Omega c d\mu'_n \right) \\
\leq \int_\Omega \varphi_n d\mu_n - \Psi \left( \int_\Omega c d\mu_n \right) + \Psi \left( \int_\Omega c d\mu'_n \right) \\
= \int_{\Omega \setminus A_n} \varphi_n d\mu_n - \Psi'(\theta) \int_{\Omega \setminus A_n} c d\mu_n \leq \int_{\Omega \setminus A_n} (\varphi_n - c) d\mu_n \leq 0,
\]
for some $\theta \in (\int_\Omega c d\mu'_n, \int_\Omega c d\mu_n)$. We can thus replace $(\varphi_n, \mu_n)$ with the pair $(\varphi'_n, \mu'_n)$, which satisfies the additional condition (5.3).
By (5.3), since \( c(x) \geq c_0 > 0 \), the total mass \( \mu_n(\Omega) \) of all these measures remains uniformly bounded. Indeed, the payoff is (see (2.7))

\[
J(\varphi_n, \mu_n) = \int_{\Omega} \varphi_n d\mu_n - \Psi \left( \int_{\Omega} c d\mu_n \right) \leq \int_{\Omega} \varphi_n d\mu_n - \Psi \left( c_0 \mu_n(\Omega) \right) = \int_{\Omega} g(x, \varphi_n) dx - \Psi \left( c_0 \mu_n(\Omega) \right) \leq Mm_N(\Omega) - \Psi \left( c_0 \mu_n(\Omega) \right).
\]

Therefore, \( J(\varphi_n, \mu_n) \to -\infty \) if \( \mu_n(\Omega) \to \infty \).

Consider the measures \( \nu_n = \varphi_n \mu_n \). By possibly taking a subsequence, we can assume (see (2.6), (2.8), and (5.2))

\[
\nu_n \rightharpoonup \nu, \quad \mu_n \rightharpoonup \mu \quad \text{in the sense of measures in } \Omega,
\]

and

\[
\varphi_n \to \varphi \quad \text{strongly in } L^p(\Omega), \quad 1 \leq p < \infty, \text{ and a.e. in } \Omega,
\]

\[
\varphi_n \rightharpoonup \varphi \quad \text{weakly in } H^1(\Omega),
\]

for some non-negative Radon measure \( \nu \) and some upper semicontinuous function \( \varphi \) which provides a weak solution to

\[
\begin{align*}
\Delta \varphi + g(x, \varphi) &= \nu, \quad x \in \Omega, \\
\nabla \varphi \cdot n &= 0, \quad x \in \partial \Omega.
\end{align*}
\]

We now define the strategy as \((\varphi^*, \mu^*)\) by setting

\[
(5.5) \quad \varphi^* \doteq \varphi, \quad \mu^* \doteq \frac{\nu}{\varphi}.
\]

By (5.3) we have

\[
(5.6) \quad \text{supp}(\nu) \subset \{ x \in \overline{\Omega}; \varphi(x) \geq c_0 \},
\]

hence the above definition is meaningful. Moreover, by (5.5)-(5.6) it follows that

\[
\Delta \varphi^* + g(x, \varphi^*) = \varphi^* \mu^*.
\]

Clearly we have also

\[
(5.7) \quad \text{supp}(\mu^*) \subset \{ x \in \overline{\Omega}; \varphi(x) \geq c_0 \}.
\]

We now establish the key inequality

\[
(5.8) \quad \mu^* \leq \mu.
\]

We have to prove that

\[
(5.9) \quad \int_{\Omega} \phi d\mu^* \leq \int_{\Omega} \phi d\mu, \quad \text{for every } \phi \in C(\overline{\Omega}), \phi \geq 0.
\]

Let \( \phi \in C(\overline{\Omega}), \phi \geq 0 \). Since \( \frac{1}{\varphi} \in H^1(\{ \varphi \geq c_0 \}) \) there exists a sequence of functions \( \eta_k \in C(\overline{\Omega}), \eta_k \geq 0 \), such that

\[
\eta_k \to \frac{1}{\varphi} \quad \text{strongly in } L^p(\{ \varphi \geq c_0 \}), \quad 1 \leq p < \infty, \text{ and a.e. in } \{ \varphi \geq c_0 \},
\]

\[
\eta_k \rightharpoonup \frac{1}{\varphi} \quad \text{weakly in } H^1(\{ \varphi \geq c_0 \}).
\]

Thanks to Lemma 4.1, for every \( k \geq 1 \) we have

\[
\limsup_{n \to \infty} \int_{\Omega} \phi \eta_k \varphi_n d\mu_n \leq \int_{\Omega} \phi \eta_k \varphi d\mu,
\]

and from (5.4), (5.5)

\[
\lim_{n \to \infty} \int_{\Omega} \phi \eta_k \varphi_n d\mu_n = \lim_{n \to \infty} \int_{\Omega} \phi \eta_k d\nu_n = \int_{\Omega} \phi \eta_k d\nu = \int_{\Omega} \phi \eta_k \varphi d\mu^*.
\]
therefore
\[
\int_{\Omega} \phi_k \varphi d\mu^* \leq \int_{\Omega} \phi_k \varphi d\mu.
\]
Letting \( k \to \infty \), by (5.6) and (5.7) we obtain (5.9).

We conclude by proving that \((\varphi^*, \mu^*)\) is an optimal strategy. Since \((\varphi_n, \mu_n)_{n \geq 1}\) is a maximizing sequence, using (5.4) and (5.8) we obtain
\[
\sup J = \lim_{n \to \infty} J(\varphi_n, \mu_n) = \lim_{n \to \infty} \left( \int_{\Omega} \varphi_n d\mu_n - \Psi \left( \int_{\Omega} c d\mu_n \right) \right)
\]
\[
= \lim_{n \to \infty} \left( \int_{\Omega} \varphi^* d\mu^* - \Psi \left( \int_{\Omega} c d\mu \right) \right) = \int_{\Omega} \varphi^* d\mu^* - \Psi \left( \int_{\Omega} c d\mu \right) = J(\varphi^*, \mu^*).
\]
The lower semicontinuity of \( b(\cdot) \) yields
\[
\int_{\Omega} b d\mu^* \leq \lim_{n \to \infty} \int_{\Omega} b d\mu_n \leq \limsup_{n \to \infty} \int_{\Omega} b d\mu_n \leq 1.
\]
This shows that the pair \((\varphi^*, \mu^*)\) is admissible, completing the proof. \( \Box \)

**Remark 5.1.** The optimal control measure satisfies
\[
\mu(\Omega_c) = 0, \quad \text{where} \quad \Omega_c = \left\{ x \in \Omega ; \varphi(x) < c(x) \right\}.
\]
Indeed, if the above relation does not hold, we can replace \( \mu \) by a new measure \( \mu'(A) = \mu(A \cap \Omega_c) \) and strictly increase the total payoff.

In particular, if \( c(x) \geq c_0 > 0 \) and \( h(x) \geq c_0 \) for all \( x \in \Omega \), then the optimal fish population satisfies
\[
\varphi(x) \geq c_0 \quad x \in \Omega.
\]

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