FRONT TRACKING APPROXIMATIONS FOR SLOW EROSION

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(Communicated by the associate editor name)

Abstract. In this paper we study an integro-differential equation describing slow erosion, in a model of granular flow. In this equation the flux is non local and depends on $x$, $t$. We define approximate solutions by using a front tracking technique, adapted to this special equation. Convergence of the approximate solutions is established by means of suitable a priori estimates. In turn, these yield the global existence of entropy solutions in BV. Such entropy solutions are shown to be unique.

We also prove the continuous dependence on initial data and on the erosion function, for the approximate as well as for the exact solutions. This establishes the well-posedness of the Cauchy problem.

1. Introduction and preliminaries. Consider the scalar integro-differential equation

$$u(t, x) + \left( f(u(t, x)) \exp \left\{ \int_x^\infty f(u(t, s)) \, ds \right\} \right)_x = 0,$$

(1)

associated with the initial data

$$u(0, x) = \bar{u}(x), \quad x \in \mathbb{R}.$$  

(2)

This equation was first derived in [2] as the slow erosion limit for a granular flow model proposed in [16], with a specific function $f$. A more general model was later derived in [23, 3], for a wider class of functions $f$. Here, the unknown variable $u$ describes the slope of the standing profile of granular matter, which varies in time due to the occurrence of small avalanches. The function $f$ is called the erosion function, which denotes the erosion rate per unit length in space covered by the avalanche. See [23] for a more detailed derivation of the model.

To simplify notation, we let $F$ denote the integral term, i.e.,

$$F(x; u) = \exp \left\{ \int_x^\infty f(u(t, s)) \, ds \right\},$$

(3)

and we write (1) as

$$u_t + (fF)_x = 0.$$ 

(4)

2000 Mathematics Subject Classification. Primary: 35L65; Secondary: 35L67, 35Q70, 35L60, 35L03.

Key words and phrases. Granular flow, slow erosion, conservation laws, front tracking, existence and uniqueness of BV solutions.

The second author is partially supported by NSF grant DMS-0908047.
Throughout this paper, we assume that the erosion function $f \in C^2((0, +\infty))$ satisfies the conditions (F):

$$f(1) = 0, \quad f' > 0, \quad f'' < 0, \quad \lim_{s \to 0^+} f(s) = -\infty, \quad \lim_{s \to +\infty} f(s)/s = 0. \quad (5)$$

The physical meaning of these assumptions is as follows. (i) At the critical slope $u = 1$ there is no erosion or deposition, hence $f(1) = 0$. (ii) When the slope approaches 0, there is infinitely large deposition. (iii) When the slope is very large, the erosion function $f$ grows slower than any linear functions. The class of functions satisfying the assumptions (F) includes the logarithm function, as well as $f(s) = s^{a-1}$, for $0 < a < 1$.

We remark that the assumptions in (5) are sharp, in order to prevent blowup of $u$. In [23] it is proved that the slope $u$ can become unbounded in finite time, if $f(s)$ approaches a linear asymptote with positive slope, as $s \to +\infty$.

Throughout the paper we will use $\|\cdot\|_{L^1}$, $\|\cdot\|_{L^\infty}$ and $\text{TV}\{\cdot\}$ to denote the $L^1$ norm, the $L^\infty$ norm and the total variation, respectively, all in the space variable. We use sign$(\cdot)$ to denote the sign function, and $C$ to denote a generic bounded constant that does not depend on the critical variables.

Solutions of the Cauchy problem will be obtained within the class $W$ consisting of all functions $w : \mathbb{R} \to \mathbb{R}$ satisfying the property

$$\inf_x w(x) > 0, \quad w(\cdot) \in BV(\mathbb{R}), \quad \text{supp}\{w(\cdot) - 1\} \text{ is bounded}. \quad (6)$$

A definition of entropy weak solutions for (1)-(2) is now given.

**Definition 1.1.** A function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is called an entropy weak solution for (1) with initial data $\bar{u}(x) \in W$ if

- $[0,T] \ni t \to u(t, \cdot) \in W$, is continuous in $L^1_{\text{loc}}$ and $u(0, \cdot) = \bar{u}$.
- For every test function $\phi \in C^\infty(\mathbb{R}^2)$, one has the integral identity
  $$\int_0^T \int_\mathbb{R} (u \phi_t + f(u)F(x; u) \phi_x) \, dx \, dt = \int_\mathbb{R} \left( u(T, x) \phi(T, x) - u(0, x) \phi(0, x) \right) \, dx. \quad (7)$$
- For a.e. $x < y$, there exists some constant $C$ (that does not depend on $x, y$ or $t$), such that
  $$u(t, x+) - u(t, y+) \leq C \max \{1/t, 1\} (y - x). \quad (7)$$

**Remark 1.** Here the definition of entropy solution follows from that of scalar conservation law with variable coefficient

$$u_t + (k(t, x)f(u))_x = 0 \quad (8)$$

where the flux function $f$ is concave and the coefficient $k$ is positive and Lipschitz continuous. For (8), the Oleinik inequality (7) would guarantee the uniqueness of the solution, by estimates for an adjoint problem, see [18]. In our case, such duality argument runs into difficulty due to the integral term (3). However, we still use the same definition, since this inequality guarantees the Lax admissibility of shocks, and will be important in the analysis of the large time behavior of solutions. An alternative definition could have been given in terms of Kruzkov entropy inequalities, as in [3].

Existence of global BV solutions and continuous dependence on initial data for a initial-boundary value problem for (1), were studied in [3]. In that paper, an iteration technique was used, freezing the global term at every time step.
In this paper, we introduce a different approximation technique, based on wave-front tracking. Our approximate solutions are piecewise constant in space and evolve continuously in time. A precise description of the algorithm is given in Section 3.1.

A somewhat similar algorithm is used in [23] where a Hamilton-Jacobi equation (with an integral term) for the height of the profile is treated, and piecewise affine (possibly discontinuous) approximate solutions are constructed. Such front tracking algorithms provide better intuition and control over wave interactions, and allow us to derive a-priori estimates in a more straightforward way. The first part of our analysis will establish the global existence and uniqueness of BV solutions.

**Theorem 1.2.** Let $T > 0$ and an initial data $\bar{u} \in W$ be given. Then the Cauchy problem (1)-(2) admits a unique entropy weak solution $u = u(t, x)$ defined for all $t \in [0, T]$, that moreover satisfies

$$\inf_{x} u(t, x) \geq \inf_{x} \bar{u}(x), \quad \|u(t, \cdot) - 1\|_{L^1} \leq \|\bar{u} - 1\|_{L^1}.$$

The existence of solutions is proved by compactness on the family of front tracking approximate solutions. Then the uniqueness is proved using the Oleinik estimate (7), combined with an error formula for the $L^1$ distance of two different solutions, corresponding to the same initial data.

Then we study the continuous dependence of the solutions, on both the initial data and the erosion function $f$. By directly comparing the $L^1$ distance between two piecewise constant approximate solutions, passing to the limit we obtain the continuous dependence for the exact solutions. By uniqueness (stated in Theorem 1.2), the same conclusion holds for any pair of entropy solutions.

More precisely, consider a second Cauchy problem

$$v_t + \left(g(v) \exp \left\{ \int_{x}^{\infty} g(v(t, s)) \, ds \right\} \right)_x = 0, \quad v(0, x) = \bar{v}(x) \in W \quad (9)$$

with an erosion function $g$ that satisfies the same assumptions (F), i.e,

$$g(1) = 0, \quad g' > 0, \quad g'' < 0, \quad \lim_{s \to 0} g(s) = -\infty, \quad \lim_{s \to +\infty} \frac{g(s)}{s} = 0. \quad (10)$$

Note that it is important to have the same critical slope for both erosion functions, otherwise $\|\bar{u} - \bar{v}\|_{L^1}$ would already be unbounded.

We have the following stability Theorem.

**Theorem 1.3.** Let $u$ and $v$ be entropy weak solutions for the Cauchy problems (1)-(2) and (9), respectively. Denote by $\kappa_0$ and $M$ a lower and upper bound for both $u$ and $v$, respectively, and by $m_0$ a bound on $\|\bar{u} - 1\|_{L^1}, \|\bar{v} - 1\|_{L^1}$.

Then

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1} \leq \|\bar{u} - \bar{v}\|_{L^1} + C \int_0^t \|u(s, \cdot) - v(s, \cdot)\|_{L^1} \, ds$$

$$+ C t \left[ \|f - g\|_{L^\infty} + \|f - g\|_{Lip} \right] \quad (11)$$

where

$$\|f - g\|_{L^\infty} \doteq \max_{\kappa_0 \leq s \leq M} |f(s) - g(s)|,$$

$$\|f - g\|_{Lip} \doteq \max_{\kappa_0 \leq s \leq M} |f'(s) - g'(s)|,$$

and $C$ depends only on $\kappa_0, M, m_0$. 

By Gronwall’s Lemma, (11) gives continuous dependence.

Other PDE models for granular flow can be found in [14, 20, 4, 11]. For mathematical properties of the steady state solutions we refer to [9, 10]. A numerical study can be found in [15]. For time-dependent solutions, see the recent results [22, 1, 2, 3, 23]. Other well-known examples of conservation law involving integral terms include the Camassa-Holm equation [8, 6] and a variational wave equation [7]. For some related results on stability for general scalar balance law, we refer to [17, 12].

The rest of the paper is structured in the following way. In Section 2 we give the basic analysis and some formal arguments. In Section 3 we prove Theorem 1.2 by front tracking approximation and uniqueness argument. Finally, in Section 4 we prove Theorem 1.3.

2. Basic analysis. By the method of characteristics, for smooth solutions one has

\[ \dot{x} = f'(u)F, \quad \dot{u} = u_t + \dot{x}u_x = f^2(u)F. \]  

(12)

Due to the nonlinearity of the erosion function \( f \), characteristics will merge, which leads to discontinuities in solutions. We call them shocks or shock waves. To see how these shocks form, let \( z = u_x \), and consider its evolution along the characteristic,

\[ \dot{z} = z_t + f'F z_x = -f''F z^2 + 3f'F z - f^3 F. \]  

(13)

Assuming \( u, f, f' \) bounded, the first term \(-f''F z^2\) dominates as \(|z|\) is large. Since \(-f''F > 0\), then \( z \) blows up to \(+\infty\) in finite time, leading to an upward jump in \( u \).

The traveling speed of the shock waves satisfies the Rankine-Hugoniot jump condition. Let \( u \) has a jump at \( x_0 \), with \( u(x^-_0) = u^- \) and \( u(x^+_0) = u^+ \). The Rankine-Hugoniot condition gives

\[ \lambda_s = F(x_0; u)^{\frac{f(u^-) - f(u^+)}{u^- - u^+}}. \]  

(14)

Since \( f \) is concave, only upward jumps are admissible. Initial downward jumps will open up into rarefaction waves. This is confirmed by (13), where \( z \) blows up only to \(+\infty\), i.e.,

\[ z \geq -C \max\{1/t, 1\}. \]  

(15)

Therefore, an Oleinik-type one-sided entropy inequality (see [19]) holds: for any \( t > 0 \), and \( x < y \), one has

\[ u(t, x) - u(t, y) \leq (y - x)C \max\{1/t, 1\}. \]  

(16)

Wave interactions are determined by the local behavior of the flux, i.e., the erosion function \( f \), which is a concave function. The interactions are similar to those of a scalar conservation law. When two (or more) admissible shocks interact, they will simply merge into a bigger admissible shock. No new waves would be formed at interactions.

Next is a technical Lemma connecting properties of \( u \) with the global term \( F \).

Lemma 2.1. Let \( u : \mathbb{R} \to \mathbb{R} \) satisfy

\[ u(x) \geq \kappa_0 > 0, \quad \|u(\cdot) - 1\|_{L^1} \leq m_0. \]

Then, the function \( f(u(x)) \) is absolutely integrable, i.e.,

\[ \|f(u(\cdot))\|_{L^1} \leq \int_{\mathbb{R}} |f(u(x))| \, dx = C \leq f'(-\kappa_0) m_0. \]  

(17)
Furthermore, the integral function $F(x; u)$ as defined in (3) satisfies

$$e^{-C} \leq F \leq e^{C}, \quad TV\{F\} \leq Ce^C. \tag{18}$$

**Proof.** Since $\|u - 1\|_{L_1} \leq m_0$, then the function $x \mapsto f(u(x))$ is absolutely integrable, because $u \mapsto f(u)$ is uniformly Lipschitz on $[\kappa_0, \infty]$ and $f(1) = 0$. Moreover

$$\int_\mathbb{R} |f(u(x))| \, dx = \int_\mathbb{R} |f(u(x)) - f(1)| \, dx \leq f'(\kappa_0) \|u - 1\|_{L_1}.$$  

This gives (17). The upper and lower bound on $F$ is obvious by its definition and (17). Finally, since $x \mapsto F$ is Lipschitz continuous, we have

$$TV\{F\} = \|F_x\|_{L_1} = \|f(u)F\|_{L_1} \leq \|F\|_{L_\infty} \|f(u)\|_{L_1} \leq Ce^C. \tag{19}$$

Below we give some formal arguments, which serves as guideline for the a priori estimates for the approximate solutions.

1. **Lower bound on $u$.** By (12), $u$ is non-decreasing along characteristics, therefore the lower bound follows.

2. **Bound on total mass.** The trivial solution is $u \equiv 1$. Equation (1) can be written as

$$(u - 1)_t + (f(u)F)_x = 0. \tag{20}$$

By the assumptions (5) we have $\text{sign}(u - 1) = \text{sign}(f(u))$. Since $F > 0$, we conclude that the $L^1$ norm of $u - 1$ is non-increasing in time.

By Lemma 2.1, $F$ is uniformly bounded from below and above, and has bounded variation.

3. **Bounded support for $u - 1$.** By the lower bound on $u$, the characteristic speed $f'(u)F$ is now bounded. Therefore, for $t \leq T$, the support for $u - 1$ is bounded.

4. **Upper bound on $u$.** Integrate the conservation law (20) over the region $(t, y)$ with $0 \leq t \leq T$ and $y \leq x(t)$ where $t \to x(t)$ is a characteristic, we get

$$\left| \int_0^T [(u - 1)f'(u) - f(u)] F \, dt \right| = \left| \int_{-\infty}^{x(T)} (u(T, x) - 1) dx - \int_{-\infty}^{x(0)} (u(0, x) - 1) dx \right| \leq 2m_0, \tag{21}$$

thanks to the bound on $\|u - 1\|_{L_1}$. Define an auxiliary function

$$\alpha(u) \equiv \frac{u - 1}{f(u)} \quad \text{if } u \neq 1, \quad \alpha(1) = 1/f'(1). \tag{22}$$

This function is well-defined for all $u > 0$. At $u = 0$ we can set $\alpha(0) = 0$ by continuity. The function is nonnegative, $\alpha(u) > 0$ for $u > 0$, and is increasing in $u$, i.e.,

$$\alpha'(u) = \frac{f(u) - (u - 1)f'(u)}{f^2(u)} > 0 \quad \text{for } u > 0, \ u \neq 1. \tag{23}$$

By the last assumption on $f$ in (5), $\alpha(u)$ grows to $+\infty$ as $u \to +\infty$,

$$\lim_{u \to +\infty} \alpha(u) = +\infty. \tag{24}$$

The evolution of $\alpha(u)$ along a characteristic is

$$\frac{d}{dt} \alpha(u(t, x(t))) = \alpha'(u(t)) \dot{u} = [f(u) - (u - 1)f'(u)] F.$$
By (21), we have, for all $T$,

$$\alpha(u(T, x(T))) \leq \alpha(u(0, x(0))) + 2m_0.$$  

By (24) we conclude that $u(t, x)$ remains bounded for all $t, x$.

(5). BV bound on $u$. Let $z = u_x$. Differentiating (1) in $x$, one gets

$$z_t + (f'(u)F(x; u)z)_x = (f^2(u)F(x; u))_x.$$  

Let $\kappa_0 > 0$ and $M$ be a lower and upper bound for $u$, respectively. We define

$$||f||_{L^\infty} \equiv \max_{\kappa_0 \leq s \leq M} |f(s)|.$$  

Formally we have

$$\frac{d}{dt} TV\{u\} \leq TV\{f^2(u)F\} \leq ||f||^2_{L^\infty} TV\{F\} + 2 ||f||_{L^\infty} f'(\kappa_0)TV\{u\} ||F||_{L^\infty} \leq C(1 + TV\{u\}).$$  

Therefore, $TV\{u\}$ can grow exponentially, but remains bounded for finite time.

3. Front tracking approximate solutions. In this section we prove Theorem 1.2. The algorithm for the piecewise constant front tracking approximation is described in Section 3.1. Then we establish a priori estimates in Section 3.2. All estimates are used in Section 3.3 to achieve compactness, which leads to existence of entropy weak solutions. Finally, uniqueness of entropy weak solutions is treated in Section 3.4, completing the proof.

3.1. The algorithm. Let $\varepsilon$ be the approximation parameter, and $u^\varepsilon$ be the $\varepsilon$-approximate solution that we now construct. For a given initial data $\bar{u} \in W$, one can construct a piecewise constant approximation, call it $\bar{u}^\varepsilon$, such that $\bar{u}^\varepsilon \rightarrow \bar{u}$ in $L^{1}_{loc}$, and $\bar{u}^\varepsilon \in W$. The approximation could be achieved by a suitable sampling in $\bar{u}$. This will be the discrete initial data for the algorithm, i.e., $u^\varepsilon(0, x) = \bar{u}^\varepsilon(x)$ and choose it such that $\bar{u}^\varepsilon(x) \leq \bar{u}(x)$. Let $m_0, \kappa_0$ satisfy

$$u^\varepsilon(0, x) \geq \kappa_0 > 0 \quad \forall \ x, \quad ||\bar{u}^\varepsilon - 1||_{L^1} \leq m_0, \quad \varepsilon > 0.$$  

Let $x_i$ ($i = 0, \cdots , N$) be the points where $u^\varepsilon$ has jumps, which we call fronts, and write

$$u_{i+\frac{1}{2}}(t) = u^\varepsilon(t, x) \quad \text{for} \ x \in [x_i, x_{i+1}).$$  

The algorithm will result in a set of ODEs that govern the evolution of $x_i$ and $u_{i+\frac{1}{2}}$ in $t$.

The approximation to the initial data must satisfy the following requirements.

- The downward jumps should be small because they are not admissible. Introduce the quantities

$$\eta_i(t) = u_{i-\frac{1}{2}}(t) - u_{i+\frac{1}{2}}(t), \quad \eta(t) = \max_{i} \eta_i(t).$$  

Note that $\eta(t)$ measures the size of the downward jumps at $t$. We require that

$$\eta(0) \leq \varepsilon.$$  

This ensures that possible initial (big) downward jumps will open up into a fan of small downward jumps, each one of size $\leq \varepsilon$. We refer to downward jumps as rarefaction fronts, and upward jumps as shock fronts.

- Whenever $\bar{u}$ crosses 1 with negative gradient, we will make sure that $u = 1$ is sampled. This will lead to a clean a priori $L^1$ estimate.
Remark 2. The choice of \( \dot{u}_{i+\frac{1}{2}} \) in (35) is motivated as follows. In order to keep \( u^\varepsilon \) constant on the interval \([x_i, x_{i+1}]\), \( u^\varepsilon_i \) must be piecewise constant. This leads to a piecewise constant approximation for \( F^\varepsilon_i \), by a finite difference of the form

\[
F^\varepsilon_i(x) \approx \frac{F^\varepsilon(x_{i+1}) - F^\varepsilon(x_i)}{x_{i+1} - x_i}, \quad x \in [x_i, x_{i+1}].
\]  

In the end, the \( \varepsilon \)-approximate solution \( u^\varepsilon \) is a weak solution of the approximate equation

\[
u^\varepsilon_i + (f(u^\varepsilon) \tilde{F}^\varepsilon)_x = 0,
\]  

where for every given \( t \), \( \tilde{F}^\varepsilon \) is a linear interpolation of \( F^\varepsilon \) in \( x \) through nodal points, i.e.,

\[
\tilde{F}^\varepsilon(x) \triangleq F^\varepsilon(x_i) \frac{x_{i+1} - x}{x_{i+1} - x_i} + F^\varepsilon(x_{i+1}) \frac{x - x_i}{x_{i+1} - x_i}, \quad \text{for } x \in [x_i, x_{i+1}].
\]
Remark 3. A front may change type in time, even without interaction. This is different from standard front tracking approximation for conservation laws. Indeed, we have
\[
\dot{u}_{i-\frac{1}{2}}(t) - \dot{u}_{i+\frac{1}{2}}(t) = -f(u_{i-\frac{1}{2}}) \frac{F^e(x_i) - F^e(x_{i-1})}{x_i - x_{i-1}} + f(u_{i+\frac{1}{2}}) \frac{F^e(x_{i+1}) - F^e(x_i)}{x_{i+1} - x_i} = F^e(x_i) \left\{ f^2(u_{i-\frac{1}{2}}) e^r - f^2(u_{i+\frac{1}{2}}) e^s \right\}
\]
for some values \( r \) and \( s \) that satisfy \( \text{sign}(r) = \text{sign}(u_{i-\frac{1}{2}} - 1) \), \( \text{sign}(s) = \text{sign}(1 - u_{i+\frac{1}{2}}) \) and \(|r| \leq \zeta_{i-\frac{1}{2}} \), \(|s| \leq \zeta_{i+\frac{1}{2}} \). Front type can change in the following cases.

- For a rarefaction front (i.e., a downward jump) with \( 1 > u_{i-\frac{1}{2}} > u_{i+\frac{1}{2}} \), we have \( \dot{\eta}_i(t) < 0 \). This front may evolve into a shock front with \( 1 > u_{i+\frac{1}{2}} > u_{i-\frac{1}{2}} \).
- For a shock front (i.e., an upward jump) with \( 1 < u_{i-\frac{1}{2}} < u_{i+\frac{1}{2}} \), it may change into a rarefaction front.

Remark 4. Two rarefaction fronts would never approach each other. If two nearby fronts approach, say \( x_i(t) = x_{i+1}(t) \), then one of them must be a shock front, and we must have
\[
\dot{x}_i(t) \geq \dot{x}_{i+1}(t) \Rightarrow u_{i-\frac{1}{2}}(t) \leq u_{i+\frac{1}{2}}(t),
\]
so the out-coming front must be a shock front. If more than two fronts merge simultaneously, say \( x_i(t) = \cdots = x_j(t) \) with \( i < j - 1 \), then between each two rarefaction fronts there must be at least one shock front. We can pair each rarefaction front with a neighboring shock front, possibly leaving the first front unpaired. By the discussion above, each pair must result in a shock front. Then, if the unpaired front \( x_i \) is a rarefaction, the final out-coming front could be a rarefaction, but with size smaller than that of \( x_i \). This implies that the maximum size of rarefaction fronts \( \eta(t) \) does not increase at any merging time.

3.2. A priori estimates. All the a priori estimates are summarized in the next Lemma.

Lemma 3.1. Let \( u^\varepsilon \) be an \( \varepsilon \)-approximate solution with initial data \( \tilde{u}^\varepsilon \in \mathcal{W} \) that satisfies (28) and (31). Then, for any \( t \in [0, T] \), we have \( x \to u^\varepsilon(t, x) \in \mathcal{W} \). For \( \varepsilon \) sufficiently small we have
\[
\eta(t) \leq C \varepsilon, \quad \zeta(t) \leq C \varepsilon,
\]
for some constant \( C \) independent of \( \varepsilon \).

Proof. (1). Lower bound for \( u^\varepsilon \). By (37) we clearly have \( \dot{u}_{i+\frac{1}{2}} \geq 0 \). The lower bound follows,
\[
\inf_x u^\varepsilon(t, x) \geq \inf_x u^\varepsilon(0, x) \geq \kappa_0.
\]

(2). Bound on \( \| u^\varepsilon - 1 \|_{L_1} \). This follows from the facts that all fronts travel with Rankine-Hugoniot speed and \( u = 1 \) is always sampled when \( u^\varepsilon \) crosses 1 with negative slope. In more detail, since \( u^\varepsilon \) is piecewise constant, we have
\[
\| u^\varepsilon - 1 \|_{L_1} = \sum_i \left| u_{i+\frac{1}{2}} - 1 \right| (x_{i+1} - x_i).
\]
A direct computation, by using summation-by-parts, gives
\[
\frac{d}{dt} \|u^\varepsilon - 1\|_{L^1} = \sum_i \text{sign}(u_{i+\frac{1}{2}} - 1)u_{i+\frac{1}{2}}(x_{i+1} - x_i) + \left| u_{i+\frac{1}{2}} - 1 \right| (\dot{x}_{i+1} - \dot{x}_i)
\]
\[
= \sum_i F^\varepsilon(x_i)I_i,
\]
where
\[
I_i = \left| f(u_{i+\frac{1}{2}}) - f(u_{i-\frac{1}{2}}) \right| + \frac{f(u_{i-\frac{1}{2}}) - f(u_{i+\frac{1}{2}})}{u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}} \left( \|u_{i-\frac{1}{2}} - 1\| - \|u_{i+\frac{1}{2}} - 1\| \right).
\]

There are several situations.

- If \( \text{sign}(u_{i-\frac{1}{2}} - 1) = \text{sign}(u_{i+\frac{1}{2}} - 1) \), then \( I_i = 0 \);
- If \( \text{sign}(u_{i-\frac{1}{2}} - 1) \neq \text{sign}(u_{i+\frac{1}{2}} - 1) \) and \( u_{i-\frac{1}{2}} \geq 1 \geq u_{i+\frac{1}{2}} \), then by construction we must have either \( u_{i+\frac{1}{2}} = 1 \) or \( u_{i-\frac{1}{2}} = 1 \). In either case we have \( I_i = 0 \);
- If \( \text{sign}(u_{i-\frac{1}{2}} - 1) \neq \text{sign}(u_{i+\frac{1}{2}} - 1) \) and \( u_{i-\frac{1}{2}} \leq 1 \leq u_{i+\frac{1}{2}} \), then by concavity of \( f \) we have
\[
\left| f(u_{i+\frac{1}{2}}) \right| \leq \frac{f(u_{i-\frac{1}{2}}) - f(u_{i+\frac{1}{2}})}{u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}} \left| u_{i+\frac{1}{2}} - 1 \right|,
\]
\[
\left| f(u_{i-\frac{1}{2}}) \right| \geq \frac{f(u_{i-\frac{1}{2}}) - f(u_{i+\frac{1}{2}})}{u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}} \left| u_{i-\frac{1}{2}} - 1 \right|.
\]

Therefore \( I_i \leq 0 \).

In conclusion, we have for all \( t \geq 0 \),
\[
\frac{d}{dt} \|u^\varepsilon(t, \cdot) - 1\|_{L^1} \leq 0, \quad \Rightarrow \quad \|u^\varepsilon(t, \cdot) - 1\|_{L^1} \leq \|u^\varepsilon(0, \cdot) - 1\|_{L^1} \leq m_0.
\]

Furthermore, by Lemma 2.1, the integral term \( F^\varepsilon \) satisfies
\[
e^{-C} \leq F^\varepsilon \leq e^C, \quad TV(F^\varepsilon) \leq C e^C,
\]
where
\[
C \doteq f'(\kappa_0)m_0 \geq \sup_t \|f(u^\varepsilon(t, \cdot))\|_{L^1}.
\]

3. **Bound on the support for \( u^\varepsilon - 1 \).** This is obvious since the speeds for the first and last fronts are bounded, thanks to the lower bound on \( u^\varepsilon \).

4. **A priori bounds on \( \eta \) and \( \zeta \).** Assume a priori that, for some \( \hat{t} \leq T \) and \( M > 0 \)
\[
u_{\hat{i} + \frac{1}{2}}(t) < M, \quad \text{for} \quad t < \hat{t}, \quad \forall i.
\]

We have
\[
\dot{\eta}_i(t) = u_{\hat{i} - \frac{1}{2}}(t) - u_{\hat{i} + \frac{1}{2}}(t) = f^2(u_{\hat{i} - \frac{1}{2}})F^\varepsilon(\tilde{x}_{\hat{i} - \frac{1}{2}}) - f^2(u_{\hat{i} + \frac{1}{2}})F^\varepsilon(\tilde{x}_{i + \frac{1}{2}})
\]
\[
= \left( f^2(u_{\hat{i} - \frac{1}{2}}) - f^2(u_{\hat{i} + \frac{1}{2}}) \right) F^\varepsilon(\tilde{x}_{\hat{i} - \frac{1}{2}}) + f^2(u_{\hat{i} + \frac{1}{2}}) \left( F^\varepsilon(\tilde{x}_{\hat{i} - \frac{1}{2}}) - F^\varepsilon(\tilde{x}_{\hat{i} + \frac{1}{2}}) \right)
\]
\[
\leq C_1 M \eta + C_2 M^2 \zeta
\]
for some $C_1, C_2$ depending on $f, \kappa_0, \|F^e\|_{L^\infty}$. For $\zeta(t)$, we have
\[
\dot{\zeta}_{i+\frac{1}{2}}(t) = (x_{i+1} - x_i)\left| f(u_{i+\frac{1}{2}}) \right| + (x_{i+1} - x_i)\text{sign}(f(u_{i+\frac{1}{2}}))f'(u_{i+\frac{1}{2}})u_{i+\frac{1}{2}}
\]
\[
= (x_{i+1} - x_i)\left| f(u_{i+\frac{1}{2}}) \right| - f'(u_{i+\frac{1}{2}})\left| f(u_{i+\frac{1}{2}}) \right| \left( F^e(t, x_{i+1}) - F^e(t, x_i) \right)
\]
These two terms are negative if $x_i, x_{i+1}$ are both shock fronts. If one of them is a rarefaction front, say $u_{i+\frac{1}{2}} \leq u_{i-\frac{1}{2}}$, then we have
\[
f'(u_{i+\frac{1}{2}})F^e(t, x_i) - \dot{x}_i \leq \|F^e\|_{L^\infty} \sup_{t \leq s \leq M} |f''(s)| (u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}) \leq C(M)\eta.
\]
This implies
\[
\dot{\zeta}_{i+\frac{1}{2}} \leq C(M)M\eta,
\]
for some $C$ possibly depending on $M$. Taking supreme over all $i$ in (45) and (46), we get
\[
\dot{\eta} \leq C_1M\eta + C_2M^2\zeta, \quad \dot{\zeta} \leq C(M)M\eta.
\]
Notice that $\zeta$ is continuous when fronts merge, while $\eta$ may be discontinuous but it does not increase (see Remark 4). Hence by standard comparison argument we arrive at
\[
\eta(t) \leq C_3\varepsilon, \quad \zeta(t) \leq C_3\varepsilon, \quad \text{for all} \quad t \in [0, \bar{t}]
\]
for some $C_3 = C_3(M, T)$.

(5). Upper bound for $u^e$. We define a discrete version of the backward characteristic curve, in the spirit of Dafermos [13].

**Definition 3.2.** Let $T > 0$. On $[0, T] \times \mathbb{R}$ we define a family of $\varepsilon$-approximate characteristics as a family of continuous curves with the following properties.

(i) For every $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}$, the continuous map $[0, \bar{t}] \ni t \mapsto x(t)$ satisfies $x(t) = \bar{x}$ and
\[
\dot{x}(t) = f'(u^e(t, x+))F^e(t, x(t)) + O(\varepsilon), \quad |O(\varepsilon)| \leq C\varepsilon.
\]
where the constant $C$ does not depend on $t, \bar{t}$ or $\bar{x}$. Furthermore, the curve does not cross any nodal curve for $0 < t < \bar{t}$.

(ii) Given any two points $(\bar{t}, \bar{x}_1)$ and $(\bar{t}, \bar{x}_2)$ with $\bar{x}_1 < \bar{x}_2$, the two corresponding curves $x_1(t)$ and $x_2(t)$ do not cross, i.e., $x_1(t) < x_2(t)$ for all $t \in (0, \bar{t})$.

The existence of $\varepsilon$-approximate characteristics is obvious once we have the estimate (48). Indeed, let $(t, x)$ be a point on an $\varepsilon$-approximate characteristics. Then, if $x$ is sufficiently away from any rarefaction front, we chose $C = 0$ in (49). Otherwise, if $x$ gets very close to a rarefaction front, say $x_i$, then we can choose $\dot{x}$ to be some intermediate value between $f'(u^e(t, x+))F^e(t, x)$ and $\dot{x}_i$, such that $x(t)$ approaches $x_i(t)$ but does not cross it as $t$ decreases. Furthermore, if $x$ is between two rarefaction fronts $x_i$ and $x_{i+1}$ and it gets very close to both fronts, then we choose $\dot{x}$ to be some intermediate value between $\dot{x}_i$ and $\dot{x}_{i+1}$, such that $x(t)$ does not cross either $x_i$ or $x_{i+1}$ as $t$ decreases. Finally, if $(\bar{t}, \bar{x})$ is already on a rarefaction front, say $(\bar{t}, \bar{x}_i)$, then we can choose $\dot{x} = f'(u_{i-\frac{1}{2}})F^e(t, x(t))$ for $t \in [\bar{t} - \delta\varepsilon, \bar{t}]$ for some small values of $\delta$, such that $x(t)$ departs from $x_i(t)$ on the right as $t$ decreases. Thanks to the estimate (48), the property (49) holds.
We now rewrite (39) as
\[(u^\varepsilon - 1)_t + (f(u^\varepsilon)\tilde{F}^\varepsilon)_x = 0.\] (50)
Consider a point \((t,x)\) such that \(x_i(t) < x < x_{i+1}(t)\) for some \(i\). Let \(t \to x(t)\) be a \(\varepsilon\)-approximate characteristic curve as in Definition 3.2, and denote by \(i = i(t)\) the index for the interval \((x_i,x_{i+1})\) where the curve remains. Integrating the conservation law (50) over the region in \((t,y)\) where the last inequality holds thanks to the bound on \(\|u^\varepsilon - 1\|_{L^1}\).

Recall the auxiliary function \(\alpha(u)\) in (22). By (23) and (37), its evolution along the \(\varepsilon\)-approximate characteristic curve satisfies
\[
\frac{d}{dt}\alpha(u_{i+\frac{1}{2}}(t)) = \alpha'(u_{i+\frac{1}{2}})u_{i+\frac{1}{2}} = \left[f(u_{i+\frac{1}{2}}) - (u_{i+\frac{1}{2}} - 1)f'(u_{i+\frac{1}{2}})\right]F^\varepsilon(\bar{x}_{i+\frac{1}{2}}),
\] (52)
where \(\bar{x}_{i+\frac{1}{2}}\) is defined in (36). We have
\[
\alpha(u_{i+\frac{1}{2}}(\bar{t})) - \alpha(u_{i+\frac{1}{2}}(0)) \\
\leq \left| \int_0^{\bar{t}} \left[f(u_{i+\frac{1}{2}}) - (u_{i+\frac{1}{2}} - 1)f'(u_{i+\frac{1}{2}})\right]F^\varepsilon(\bar{x}_{i+\frac{1}{2}}) \, dt \right|.
\]
Thanks to (48), the values \(F^\varepsilon(\bar{x}_{i+\frac{1}{2}}), \tilde{F}^\varepsilon(t,x(t))\) and \(F^\varepsilon(t,x(t))\) are very close to each other, by a factor of \(C = e^{C\varepsilon}\). We now have
\[
\alpha(u_{i+\frac{1}{2}}(\bar{t})) - \alpha(u_{i+\frac{1}{2}}(0)) \\
\leq C \left| \int_0^{\bar{t}} \left[f(u_{i+\frac{1}{2}}) - (u_{i+\frac{1}{2}} - 1)f'(u_{i+\frac{1}{2}})\right]F^\varepsilon(t,x(t)) \, dt \right| \\
\leq C \left| \int_0^{\bar{t}} \left[f(u_{i+\frac{1}{2}})\tilde{F}^\varepsilon(t,x(t)) - (u_{i+\frac{1}{2}} - 1)\dot{x}(t)\right] \, dt \right| \\
+ C \left| \int_0^{\bar{t}} (u_{i+\frac{1}{2}} - 1) \left[\dot{x}(t) - f'(u_{i+\frac{1}{2}})\tilde{F}^\varepsilon(t,x(t))\right] \, dt \right| \\
\leq 2Cm_0 + C \left| \int_0^{\bar{t}} \mathcal{I}(t) \, dt \right|.
\]
In the last inequality we used (51), and for \(\mathcal{I}(t)\) we use (49) and (48) to get
\[
\mathcal{I}(t) \doteq (u_{i+\frac{1}{2}}(t) - 1) \left[\dot{x}(t) - f'(u_{i+\frac{1}{2}}(t))\tilde{F}^\varepsilon(t,x(t))\right] \\
= (u_{i+\frac{1}{2}}(t) - 1) \left[\dot{x}(t) - f'(u_{i+\frac{1}{2}}(t))F^\varepsilon(t,x(t))\right] \\
+ (\tilde{F}^\varepsilon(t,x(t)) - F^\varepsilon(t,x(t))) \\
\leq (u_{i+\frac{1}{2}} - 1) \cdot \tilde{C} \varepsilon,
\] (53)
for some \( \tilde{C} \) does not depend on \( t, x, \varepsilon \). Now, let \( M_1 \) be a finite constant such that

\[
\alpha(M_1) = \alpha \left( \sup_x u(x) \right) + 2Cm_0, \tag{54}
\]

and let \( \tilde{t} \) be the first time in \([0, T]\) that \( \alpha(u_{i+\frac{1}{2}}) = \alpha(M_1 + 1) \), so

\[
u_{i+\frac{1}{2}}(t) < M_1 + 1, \quad \text{for} \quad t < \tilde{t}. \tag{55}
\]

Now consider (44) with \( M = M_1 + 1 \). From (53) we have \(|I(t)| \leq M_1 \tilde{C} \varepsilon \) where \( \tilde{C} \) now depends on \( M_1 \). Then

\[
\alpha(M_1 + 1) \leq \alpha(M_1) + C \left| \int_0^{\tilde{t}} I(t) \, dt \right| \leq \alpha(M_1) + CM_1 \tilde{C} \varepsilon \tilde{t},
\]

which gives

\[
\tilde{t} \geq \frac{\alpha(M_1 + 1) - \alpha(M_1)}{CM_1 \tilde{C}} \cdot \frac{1}{\varepsilon} = \frac{\tilde{C}}{\varepsilon}.
\]

By choosing \( \varepsilon \) small, \( \tilde{t} \) can be arbitrarily large, leading to the upper bound for \( u^\varepsilon \) for any finite \( T \).

In turn, this gives the uniform bounds in (41) on \( \eta \) and \( \zeta \) for \( t \in [0, T] \).

(6). \( BV \) bound for \( u^\varepsilon \). For the \( \varepsilon \)-approximate solution \( u^\varepsilon \) we have

\[
\frac{d}{dt} \text{TV}\{u^\varepsilon\} = \frac{d}{dt} \sum_i \left| u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} \right| = \sum_i \text{sign}(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}) \left| u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} \right|
\]

\[
= \sum_i \text{sign}(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}) \left| f^2(u_{i+\frac{1}{2}}) F^\varepsilon(\tilde{x}_{i+\frac{1}{2}}) - f^2(u_{i-\frac{1}{2}}) F^\varepsilon(\tilde{x}_{i-\frac{1}{2}}) \right|
\]

\[
\leq \sum_i \left| f^2(u_{i+\frac{1}{2}}) - f^2(u_{i-\frac{1}{2}}) \right| |F^\varepsilon(\tilde{x}_{i+\frac{1}{2}})| + \left| f^2(u_{i-\frac{1}{2}}) \right| |F^\varepsilon(\tilde{x}_{i+\frac{1}{2}}) - F^\varepsilon(\tilde{x}_{i-\frac{1}{2}})|
\]

\[
\leq 2 \| f \|_{L^\infty} f'(\kappa_0) \| F^\varepsilon \|_{L^\infty} \text{TV}\{u^\varepsilon\} + \| f \|_{L^\infty}^2 \text{TV}\{F^\varepsilon\}
\]

\[
\leq C \cdot \text{TV}\{u^\varepsilon\} + C. \tag{56}
\]

Therefore, total variation of \( u^\varepsilon \) can grow exponentially in time, but remains bounded for finite time \( t \leq T \), completing the proof. \( \square \)

**Remark 5.** By Lemma 3.1 and the fact that fronts can only merge, the set of ODEs for \( x_i(t) \) in (34) and for \( u_{i+\frac{1}{2}}(t) \) in (35) are well-defined, generating unique approximate solutions.

**Remark 6.** The \( L^1 \) continuity in time for \( u^\varepsilon \) and \( F^\varepsilon \) follows by a standard argument, as a consequence of the a priori bounds in Lemma 3.1. We omit the details.

In next Lemma we establish the discrete version of the Oleinik inequality.

**Lemma 3.3.** A discrete version of a one-sided entropy inequality holds for \( u^\varepsilon \),

\[
u^\varepsilon(t,x^+) - u^\varepsilon(t,y^+) \leq C \max\{1/t, 1\}(y-x), \quad y > x + \varepsilon \tag{57}
\]

where the constant \( C \) does not depend on \( x, y, t \) or \( \varepsilon \).
Proof. Fix $\tau > 0$, with $\tau \leq T$, and two points $y_1, y_2$ such that $y_1 + \varepsilon < y_2$. Then there exist two $\varepsilon$-approximate characteristics $x_1(\cdot)$ and $x_2(\cdot)$ defined on $[0, \tau]$, with $x_1(\tau) = y_1, \quad x_2(\tau) = y_2, \quad x_1(t) < x_2(t) \quad \forall \ t \in [0, \tau]$. 

Recalling (37), along any $\varepsilon$-approximate characteristic curve $x(\cdot)$ one has 

$$
\frac{d}{dt} u^\varepsilon(t, x(t)) = f^\varepsilon(u^\varepsilon(t, x)) F^\varepsilon(t, x) + O(\varepsilon). \tag{58}
$$

By Lemma 3.1, there exist positive constants $L, M$ and $c$, such that $|O(\varepsilon)| \leq M\varepsilon$ and 

$$
|F^\varepsilon(t, x) - F^\varepsilon(t, y)| \leq L |x - y|, \quad f''(u) \leq -c < 0, \\
0 \leq f^2(u) \cdot F \leq M, \quad |f(u)|, |f'(u)| \leq M, \quad \frac{1}{M} \leq F \leq M.
$$

Let $u(\tau, y_1 + ) = \bar{u}_1$ and $u(\tau, y_2 + ) = \bar{u}_2$, and consider $\bar{u}_1 > \bar{u}_2$ (otherwise the estimate holds trivially). Then there exists $\tau_0$ with $0 \leq \tau_0 < \tau$ such that $u(t, x_1(t)) - u(t, x_2(t)) \geq 0$ for $t \in [\tau_0, \tau]$, possibly vanishing for $t = \tau_0$. In this time interval we have 

$$
\frac{d}{dt} (x_2(t) - x_1(t)) \\
= f'(u(t, x_2(t)) F^\varepsilon(t, x_2(t)) - f'(u(t, x_1(t)) F^\varepsilon(t, x_1(t)) + O(\varepsilon) \\
= [f'(u(t, x_2) - f'(u(t, x_1))] F^\varepsilon(t, x_2) + f'(u(t, x_1)) [F^\varepsilon(t, x_2) - F^\varepsilon(t, x_1)] + O(\varepsilon) \\
\geq \frac{c}{M} [u(t, x_1(t) - u(t, x_2(t)] - ML(x_2(t) - x_1(t)) - |O(\varepsilon)|]. \tag{59}
$$

Now assume in addition 

$$
x_2(t) - x_1(t) \leq y_2 - y_1 \quad \forall t \in [t_0, \tau],
$$

where $t_0 \geq \tau_0$ will be specified later. From (59) we deduce 

$$
\frac{d}{dt} (x_2(t) - x_1(t)) \\
\geq \frac{c}{M} [u(t, x_1(t) - u(t, x_2(t)] - ML(y_2 - y_1) - |O(\varepsilon)] \\
\geq \frac{c}{M} [u(t, x_1(t) - u(t, x_2(t)] - M_1(y_2 - y_1) \tag{61}
$$

where $M_1$ depends on $M$ and $L$. Moreover, from (58) we deduce 

$$
\frac{d}{dt} (u(t, x_1(t)) - u(t, x_2(t))) \\
\leq L_1 (u(t, x_1(t)) - u(t, x_2(t))) + M_2 (x_2(t) - x_1(t)) + O(\varepsilon) \\
\leq L_1 (u(t, x_1(t)) - u(t, x_2(t))) + M_3(y_2 - y_1) \tag{62}
$$

for some constants $L_1, M_2,$ and $M_3$ depending on $L$ and $M$. This yields 

$$
u(t, x_1(t)) - u(t, x_2(t)) \\
\geq e^{-L_1(t-t)}(\bar{u}_1 - \bar{u}_2) - M_3(y_2 - y_1)(t-t) \\
\geq e^{-L_1T} (\bar{u}_1 - \bar{u}_2) - M_3(y_2 - y_1)(t-t). \tag{63}
$$

We now consider two cases.

Case 1. If 

$$
\frac{c e^{-L_1T}}{M} (\bar{u}_1 - \bar{u}_2) \leq 4M_1 (y_2 - y_1),
$$
this immediately implies
\[ \bar{u}_1 - \bar{u}_2 \leq \frac{4M_1}{c} e^{L_1 T} (y_2 - y_1). \] (64)

**Case 2.** If instead we have
\[ \frac{c}{M} e^{-L_1 T} \cdot (\bar{u}_1 - \bar{u}_2) > 4M_1 (y_2 - y_1), \] (65)

then
\[ \frac{d}{dt}(x_2(\tau) - x_1(\tau)) > 0, \]

and by continuity there exists a time interval \([t_0, \tau]\) such that (60) holds. Indeed, we can choose \(t_0 \geq \tau_0\) such that
\[ \frac{c}{M} \left[e^{-L_1 T}(\bar{u}_1 - \bar{u}_2) - M_3(y_2 - y_1)(\tau - t)\right] \geq 3M_1(y_2 - y_1), \quad t \in [t_0, T]. \] (66)

From (63), we find that
\[ \frac{c}{3M} \left[u(t, x_1(t)) - u(t, x_2(t))\right] \geq M_1(y_2 - y_1). \]

Therefore, on this time interval, from (61) we have
\[ \frac{d}{dt}(x_2(t) - x_1(t)) \geq \frac{2c}{3M} [u(t, x_1(t)) - u(t, x_2(t))] \] (67)

hence (60) is valid on \([t_0, T]\). Integrating (67) and using (63), we now have
\[ y_2 - y_1 \geq \frac{2c}{3M} \int_{t_0}^{\tau} \left[e^{-L_1 T}(\bar{u}_1 - \bar{u}_2) - M_3(y_2 - y_1)(\tau - t)\right] dt \]
\[ \geq \frac{2c}{3M} \left\{e^{-L_1 T}(\tau - t_0)(\bar{u}_1 - \bar{u}_2) - M_3(y_2 - y_1)\frac{(\tau - t_0)^2}{2}\right\}, \]

so that
\[ \left(1 + \frac{cM_3}{3M}(\tau - t_0)^2\right)(y_2 - y_1) \geq \frac{2c}{3M} e^{-L_1 T}(\tau - t_0)(\bar{u}_1 - \bar{u}_2). \]

We require that \(t_0\) satisfies, in addition to (66),
\[ \frac{cM_3}{3M}(\tau - t_0)^2 \leq 1. \] (68)

Therefore we deduce that
\[ y_2 - y_1 \geq \frac{c e^{-L_1 T}}{3M}(\tau - t_0)(\bar{u}_1 - \bar{u}_2). \] (69)

If \(t_0 = 0\), then we have
\[ \bar{u}_1 - \bar{u}_2 \leq \frac{3Me^{L_1 T}}{c} \cdot \frac{1}{\tau} \cdot (y_2 - y_1). \]

If \(t_0 = \tau_0 > 0\), then \(u(\tau_0, x_1(\tau_0)) = u(\tau_0, x_2(\tau_0))\) and from (63) we simply have
\[ \bar{u}_1 - \bar{u}_2 \leq T e^{L_1 T} M_3(y_2 - y_1). \]

Finally if \(t_0 > \tau_0\), then the definition of \(t_0\) implies that either
\[ \tau - t_0 = \sqrt{\frac{M}{cM_3}}, \]
or else by (65)
\[ M_3(y_2 - y_1)(\tau - t_0) = e^{-\frac{L_1 T}{c}}(\bar{u}_1 - \bar{u}_2) - \frac{3MM_1}{c}(y_2 - y_1) \geq \frac{MM_1}{c}(y_2 - y_1). \]

Putting together the above two cases, we can choose a possibly larger \( t_0 \) such that
\[ \tau - t_0 = \min \left\{ \sqrt{\frac{M}{cM_3}}, \frac{MM_1}{cM_3} \right\} = M_4. \]

Since the bound above is independent on \( y_1, y_2, \varepsilon \) and \( \tau \in [0, T] \), from (69) we deduce the desired estimate
\[ \bar{u}_1 - \bar{u}_2 \leq \max \left\{ \frac{3M}{cT}, \frac{3M}{cM_4}, TM_3 \right\} e^{L_1 T}(y_2 - y_1). \] (70)

Together with (64), we complete the proof. \( \square \)

3.3. Convergence of the approximate solutions and existence of entropy weak solutions. Since all fronts \( x_i \) travel with the Rankine-Hugoniot speed, our \( \varepsilon \)-approximate solution \( u^\varepsilon \) provides a weak solution to the modified conservation law (39). Rewrite it as
\[ u^\varepsilon_t + (f(u^\varepsilon)F^\varepsilon)_x = E^\varepsilon, \quad \text{where} \quad E^\varepsilon(t, x) = \left[ f(u^\varepsilon)(F^\varepsilon - \bar{F}^\varepsilon) \right]_x. \] (71)

The following discrete weak formulation holds for all test functions \( \phi \in C_c^\infty(\mathbb{R}^2) \),
\[ \int_0^T \int_{\mathbb{R}} (u^\varepsilon \phi_t + f(u^\varepsilon)F^\varepsilon \phi_x) \, dx \, dt = \int_{\mathbb{R}} (u^\varepsilon(T, x)\phi(T, x) - u^\varepsilon(0, x)\phi(0, x)) \, dx - \int_0^T \int_{\mathbb{R}} E^\varepsilon \phi \, dx \, dt. \] (72)

To achieve existence of weak solutions, we observe that, thanks to the a priori estimates in Lemma 3.1, there exist some limit functions \( u(t, x) \) and \( F(t, x) \) such that, by extracting a subsequence \( \varepsilon \to 0 \), one has
(i) \( u^\varepsilon(0, \cdot) \to u(0, \cdot) \) and \( u^\varepsilon(T, \cdot) \to u(T, \cdot) \) in \( L^1_{loc}(\mathbb{R}) \);
(ii) \( u^\varepsilon \to u \) and \( F^\varepsilon \to F \) in \( L^1_{loc}([0, T] \times \mathbb{R}) \);
(iii) For any given \( a < b \), one has
\[ \int_a^b |E^\varepsilon(t, x)| \, dx \leq TV\{f(u^\varepsilon)\} \|F^\varepsilon - \bar{F}^\varepsilon\|_{L^\infty} + \|f\|_{L^\infty} TV\{F^\varepsilon - \bar{F}^\varepsilon\}. \]

Since \( \bar{F}^\varepsilon \) interpolates linearly \( F^\varepsilon \) through nodal points, and by Lemma 3.1, one has
\[ \|F^\varepsilon - \bar{F}^\varepsilon\|_{L^\infty} \leq C\varepsilon \]
and
\[ TV\{F^\varepsilon - \bar{F}^\varepsilon\} = \|\{F^\varepsilon - F^\varepsilon\}_x\|_{L^1} = \sum_i |f(u^\varepsilon(x_{i+1}^+)) - f(u^\varepsilon(x_{i+1}^-))| \, dx \leq \sum_i |f(u^\varepsilon(x_{i+1}^-)) - f(u^\varepsilon(x_{i+1}^+))| \, dx \leq C\varepsilon. \]
Therefore
\[ \int_a^b |E^\varepsilon(t, x)| \, dx \to 0 \]
uniformly for \( t \in [0, T] \).

(iv) Since \( u^\varepsilon \) and \( f(u^\varepsilon) \) are uniformly bounded, the identity (29) holds in the limit, i.e.,
\[ F(t, x) = \exp \left\{ \int_x^\infty f(u(t, y)) \, dy \right\} \quad \text{a.e. } (t, x) \in [0, T] \times \mathbb{R}. \]

Furthermore, by taking the limit \( \varepsilon \to 0 \) in (57), the entropy inequality holds. The existence of entropy weak solutions follows.

3.4. **Uniqueness of entropy weak solutions.** Here we complete the proof of Theorem 1.2, proving that any two entropy weak solutions of (1), corresponding to the same initial data, must coincide. Therefore, each sequence \( \{u^\varepsilon\} \) of \( \varepsilon \)-approximate solutions converges to a unique limit as \( \varepsilon \to 0 \).

Let \( T > 0 \) and \( \bar{u} \in \mathcal{W} \). Let \( u(t, x) \) be an entropy weak solution according to Definition 1.1. Hence \( u \) is the unique entropy weak solution of the problem
\[ v_t + [k(t, x)f(v)]_x = 0, \quad k(t, x) = \exp \left\{ \int_x^\infty f(u(t, s)) \, ds \right\}, \quad v(0, \cdot) = \bar{u}. \]

This fact can be proved by adapting the proof by duality argument valid for (concave) conservation laws without inhomogeneity ([19, 21]; see also [18]).

Now let \( \tilde{u}(t, x) \) be another entropy weak solution of (1) for the same initial data \( \bar{u} \). It will be the unique solution of
\[ v_t + [\tilde{k}(t, x)f(v)]_x = 0, \quad \tilde{k}(t, x) = \exp \left\{ \int_x^\infty f(\tilde{u}(t, s)) \, ds \right\}, \quad v(0, \cdot) = \bar{u}. \]

We claim that there exists a constant \( L > 0 \) such that
\[ \|u(t, \cdot) - \tilde{u}(t, \cdot)\|_{\text{L}^1(\mathbb{R})} \leq L \int_0^t \|u(\tau, \cdot) - \tilde{u}(\tau, \cdot)\|_{\text{L}^1(\mathbb{R})} \, d\tau \quad (73) \]
for all \( t \in [0, T] \). By Gronwall Lemma, this implies that \( u(t, \cdot) = \tilde{u}(t, \cdot) \).

To prove (73), we proceed as in [3, Subsect.3.3] and rely on the error formula (see [5])
\[ \|u(t, \cdot) - \tilde{u}(t, \cdot)\|_{\text{L}^1(\mathbb{R})} \leq \int_0^t E(\tau) \, d\tau \]
with
\[ E(\tau) \doteq \lim_{h \to 0^+} \sup_{\tau} \frac{\|u(\tau + h, \cdot) - u^\tau(\tau + h, \cdot)\|_{\text{L}^1(\mathbb{R})}}{h} \quad (74) \]
and where \( u^\tau(t, x) \), defined for \( t \geq \tau \), is the entropy weak solution of
\[ v_t + [\tilde{k}(t, x)f(v)]_x = 0, \quad v(\tau, x) = u(\tau, x). \]

To estimate the error term \( E(\tau) \), we proceed as in [3] by evaluating the \( \text{L}^1 \) distance of solutions to equations with different coefficients \( k \) and \( \tilde{k} \). We end up with the estimate
\[ E(\tau) \leq L \|u(\tau, \cdot) - \tilde{u}(\tau, \cdot)\|_{\text{L}^1(\mathbb{R})}. \]
where the constant $L$ depends on
\[ \|u\|_{L^\infty}, \|\tilde{u}\|_{L^\infty}, \quad TV\{u(t, \cdot)\}, \quad TV\{\tilde{u}(t, \cdot)\}, \quad TV\{k(t, \cdot)\} \]
that are all bounded uniformly in time $t \in [0, T]$. We omit the details.

4. **Stability of entropy weak solutions.** In this section we prove Theorem 1.3. Let $u, v$ be entropy weak solutions for the Cauchy problems (1)-(2) and (9), respectively, and let $\{u^\varepsilon\}_{\varepsilon > 0}$ and $\{v^\varepsilon\}_{\varepsilon > 0}$ be sequences of corresponding $\varepsilon$-approximate solutions, generated by the algorithm in Section 3.

By the uniqueness of entropy weak solutions, it will be sufficient to prove the stability of front tracking approximations. In the following, we aim at proving that
\[
\|u^\varepsilon(t, \cdot) - v^\varepsilon(t, \cdot)\|_{L^1} \leq \|\tilde{u}^\varepsilon - \tilde{v}^\varepsilon\|_{L^1} + C\int_0^t \|u^\varepsilon(s, \cdot) - v^\varepsilon(s, \cdot)\|_{L^1} \, ds + Ct \left[ \|f - g\|_{L^\infty} + \|f - g\|_{Lip} + \varepsilon \right]
\]
for a suitable constant $C$ depending only on the bounds on the solutions. Then, by taking the limit $\varepsilon \to 0^+$ in (75) and using the fact that $u^\varepsilon \to u$ and $v^\varepsilon \to v$ in $L^1_{loc}$ for all $t$, we obtain (11), proving Theorem 1.3.

Introducing the notation
\[
G(v; x) = \exp\left\{ \int_x^\infty g(v(t, y)) \, dy \right\},
\]
we can rewrite the equation for $v$ as
\[
v_t + (gG)_x = 0.
\]
Let $\kappa_0$, $M$ and $m_0$ be constants as in Theorem 1.3 for both $u^\varepsilon$ and $v^\varepsilon$, i.e.,
\[
\kappa_0 \leq u^\varepsilon(t, x) \leq M, \quad \kappa_0 \leq v^\varepsilon(t, x) \leq M, \quad \|u^\varepsilon(t, \cdot) - 1\|_{L^1} \leq m_0, \quad \|v^\varepsilon(t, \cdot) - 1\|_{L^1} \leq m_0.
\]

Recalling Lemma 2.1 and (56), the following quantities are uniformly bounded,
\[
\|F^\varepsilon\|_{L^\infty}, \quad TV\{F^\varepsilon\}, \quad TV\{u^\varepsilon\}, \quad \|f(u^\varepsilon)\|_{L^\infty}, \quad \|f(u^\varepsilon(t, \cdot))\|_{L^1}, \quad \|f(v^\varepsilon)\|_{L^\infty}, \quad \|f(v^\varepsilon(t, \cdot))\|_{L^1}.
\]

For a given $\varepsilon > 0$, let $x_i$ ($i = 0, \cdots, N$) be the points where either $u^\varepsilon$ or $v^\varepsilon$ has a jump. We have
\[
\|u^\varepsilon(t, \cdot) - v^\varepsilon(t, \cdot)\|_{L^1} \leq \sum_{i=0}^{N-1} \left| u_{i+\frac{1}{2}}(t) - v_{i+\frac{1}{2}}(t) \right| (x_{i+1}(t) - x_i(t)).
\]
Here and in the rest the summation $\sum$ is always over $i$, unless if it is stated otherwise. Differentiating (78) w.r.t. $t$, we have
\[
\frac{d}{dt} \|u^\varepsilon(t, \cdot) - v^\varepsilon(t, \cdot)\|_{L^1} = A + B
\]
where
\[
A = \sum_{i=0}^{N-1} \text{sign}(u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}})(\dot{u}_{i+\frac{1}{2}} - \dot{v}_{i+\frac{1}{2}})(x_{i+1} - x_i),
\]
\[
B = \sum_{i=0}^{N-1} \left| u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}} \right| (\dot{x}_{i+1} - \dot{x}_i).
\]
Estimates on $A$. Recall $u_{i+\frac{1}{2}}^\varepsilon$ in (35), namely
\begin{equation}
\begin{aligned}
\dot{u}_{i+\frac{1}{2}} &= -f(u_{i+\frac{1}{2}})\Phi_{i+\frac{1}{2}}, \\
\Phi_{i+\frac{1}{2}} &= \frac{F^\varepsilon(x_{i+1}) - F^\varepsilon(x_{i-1})}{x_{i+1} - x_{i-1}},
\end{aligned}
\end{equation}
where $x_{i-1}, x_{i+1}$ are the two nearby fronts for $u^\varepsilon$, and for $\dot{v}_{i+\frac{1}{2}}$,
\begin{equation}
\begin{aligned}
\dot{v}_{i+\frac{1}{2}} &= -g(v_{i+\frac{1}{2}})\Psi_{i+\frac{1}{2}}, \\
\Psi_{i+\frac{1}{2}} &= \frac{G^\varepsilon(x_{i+1}) - G^\varepsilon(x_{i-1})}{x_{i+1} - x_{i-1}},
\end{aligned}
\end{equation}
where $x_{i-1}, x_{i+1}$ are the two nearby fronts for $v^\varepsilon$. Then
\begin{equation}
\begin{aligned}
\dot{u}_{i+\frac{1}{2}} - \dot{v}_{i+\frac{1}{2}} &= -(f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}}))\Phi_{i+\frac{1}{2}} - (f(v_{i+\frac{1}{2}}) - g(v_{i+\frac{1}{2}}))\Phi_{i+\frac{1}{2}} \\
&\quad - g(v_{i+\frac{1}{2}}) (\Phi_{i+\frac{1}{2}} - \Psi_{i+\frac{1}{2}}).
\end{aligned}
\end{equation}
We can write
\begin{equation}
A = A_1 + A_2 + A_3
\end{equation}
where
\begin{equation}
A_1 = -\sum f(v_{i+\frac{1}{2}}) \left| f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}}) \right| \Phi_{i+\frac{1}{2}}(x_{i+1} - x_i),
\end{equation}
\begin{equation}
A_2 = -\sum \text{sign}(u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}})(f(v_{i+\frac{1}{2}}) - g(v_{i+\frac{1}{2}}))\Phi_{i+\frac{1}{2}}(x_{i+1} - x_i),
\end{equation}
\begin{equation}
A_3 = -\sum \text{sign}(u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}})(g(v_{i+\frac{1}{2}})(\Phi_{i+\frac{1}{2}} - \Psi_{i+\frac{1}{2}})(x_{i+1} - x_i).
\end{equation}
Note that $\Phi_{i+\frac{1}{2}}$ and $\Psi_{i+\frac{1}{2}}$ are approximations to $F^\varepsilon_x$ and $G^\varepsilon_x$ respectively on the interval $[x_i, x_{i+1})$. By our construction we have
\begin{equation}
\left| F^\varepsilon_x(x) - \Phi_{i+\frac{1}{2}} \right| \leq C\varepsilon, \quad \left| G^\varepsilon_x(x) - \Psi_{i+\frac{1}{2}} \right| \leq C\varepsilon, \quad \forall x \in [x_i, x_{i+1}).
\end{equation}
We immediately have the estimates for $A_2$ and $A_3$
\begin{equation}
A_2 \leq TV(F^\varepsilon) \cdot \|f - g\|_{L^\infty} + C\varepsilon,
\end{equation}
\begin{equation}
A_3 \leq \|g\|_{L^\infty} TV(F^\varepsilon - G^\varepsilon) + C\varepsilon.
\end{equation}
The estimate for $A_1$ will be established later in (101), together with terms from $B$. Estimates on $B$. By summation-by-parts, we have
\begin{equation}
B = \sum \left( |u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}| - |v_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}| \right) \hat{x}_i.
\end{equation}
At every $x_i$, we define the artificial speeds $s_i, \tilde{s}_i$ as follows. If $u^\varepsilon$ has a jumps at $x_i$, we let $s_i = \tilde{s}_i = \hat{x}_i$. Otherwise, if $v^\varepsilon$ has a jump at $x_i$, we let
\begin{equation}
s_i = F^\varepsilon(x_i) \cdot \frac{f(v_{i+\frac{1}{2}}) - f(v_{i-\frac{1}{2}})}{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}, \quad \tilde{s}_i = F^\varepsilon(x_i) \cdot \frac{g(v_{i+\frac{1}{2}}) - g(v_{i-\frac{1}{2}})}{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}.\end{equation}
Note that we use the $F^\varepsilon$ for the global term in all these speeds. We now have
\begin{equation}
B = B_1 + B_2 + B_3
\end{equation}
where
\begin{equation}
B_1 = \sum s_i \left( |u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}| - |v_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}| \right),
\end{equation}
\begin{equation}
B_2 = \sum (\tilde{s}_i - s_i) \left( |u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}| - |u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}| \right),
\end{equation}
\begin{equation}
B_3 = \sum (\hat{x}_i - \tilde{s}_i) \left( |u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}| - |u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}| \right).
\end{equation}
Here in $B_2$ and $B_3$ we only need to sum over all jumps in $v^\varepsilon$.

Now consider $B_1$. At every point $x_i$, we define $\lambda_i^-$ and $\lambda_i^+$ as
\[
\lambda_i^- = \frac{f(u_{i-\frac{1}{2}}) - f(v_{i-\frac{1}{2}})}{u_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}} F^\varepsilon(x_i), \quad \lambda_i^+ = \frac{f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}})}{u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}} F^\varepsilon(x_i). \quad (95)
\]
The term $B_1$ can be written as
\[
B_1 = B_{1,a} + B_{1,b},
\]
where
\[
B_{1,a} = \sum |u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}| (\lambda_i^+ - s_i) - |u_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}| (\lambda_i^- - s_i), \quad (96)
\]
\[
B_{1,b} = \sum |u_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}| \lambda_i^- - |u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}| \lambda_i^+. \quad (97)
\]
Consider $B_{1,a}$ and write $B_{1,a} = \sum B_{1,a,i}$. There are various situations. Let’s first consider if $u^\varepsilon$ has a jump at $x_i$ so $v_{i-\frac{1}{2}} = v_{i+\frac{1}{2}}$. There are several cases.

- If $u_{i-\frac{1}{2}} \geq v_{i-\frac{1}{2}}$ and $u_{i+\frac{1}{2}} \geq v_{i+\frac{1}{2}}$, we have
  \[
  B_{1,a,i} = (u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}) \lambda_i^+ - (u_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}) \lambda_i^- - (u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}) s_i = 0.
  \]

- If $u_{i-\frac{1}{2}} \leq v_{i-\frac{1}{2}}$ and $u_{i+\frac{1}{2}} \leq v_{i+\frac{1}{2}}$, it is completely similar. We have $B_{1,a,i} = 0$.

- If $u_{i-\frac{1}{2}} \leq v_{i-\frac{1}{2}} = v_{i+\frac{1}{2}} \leq u_{i+\frac{1}{2}}$, then the front is a shock. We have
  \[
  \lambda_i^+ \leq s_i \leq \lambda_i^-,
  \]
  therefore $B_{1,a,i} \leq 0$.

- If $u_{i-\frac{1}{2}} \geq v_{i-\frac{1}{2}} = v_{i+\frac{1}{2}} \geq u_{i+\frac{1}{2}}$, the front is a rarefaction, therefore it is small. We have
  \[
  \lambda_i^+ - s_i \leq C \varepsilon, \quad s_i - \lambda_i^- \leq C \varepsilon,
  \]
  therefore $B_{1,a,i} \leq C |u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}| \varepsilon$.

The cases where $v^\varepsilon$ has a jump at $x_i$ is completely similar. In summary, we have
\[
B_{1,a} \leq C \varepsilon \sum |u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}| + |v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}| \leq C \varepsilon [TV\{u^\varepsilon\} + TV\{v^\varepsilon\}]. \quad (98)
\]
For $B_{1,b}$, summation-by-parts again gives
\[
B_{1,b} = \sum |f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}})| (F^\varepsilon(x_{i+1}) - F^\varepsilon(x_i)). \quad (99)
\]
We compare $B_{1,b}$ with the term $A_1$ in (84). By construction we have (similar to (87))
\[
\left| \frac{F^\varepsilon(x_{i+1}) - F^\varepsilon(x_i)}{x_{i+1} - x_i} - \Phi_{i+\frac{1}{2}} \right| \leq C \varepsilon. \quad (100)
\]
So $B_{1,b}$ and $A_1$ are close in value, but with opposite signs. We have
\[
A_1 + B_{1,b}
\]
\[
= \sum |f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}})| (x_{i+1} - x_i) \left( \frac{F^\varepsilon(x_{i+1}) - F^\varepsilon(x_i)}{x_{i+1} - x_i} - \Phi_{i+\frac{1}{2}} \right)
\]
\[
\leq C \varepsilon \cdot \sum |f(u_{i+\frac{1}{2}}) - f(v_{i+\frac{1}{2}})| (x_{i+1} - x_i)
\]
\[
\leq C \varepsilon (\|f(u^\varepsilon(t, \cdot))\|_{L^1} + \|f(v^\varepsilon(t, \cdot))\|_{L^1})
\]
\[
\leq C \varepsilon. \quad (101)
\]
The last inequality holds because $\|f(u^\varepsilon(t, \cdot))\|_{L^1}$ and $\|f(v^\varepsilon(t, \cdot))\|_{L^1}$ are both bounded for all $t$. 

Now, consider the term $B_2$. Since we sum over all $i$ where $v^ε$ has a jump at $x_i$, we have $u_{i+\frac{1}{2}} = u_{i-\frac{1}{2}}$, therefore

$$
|u_{i-\frac{1}{2}} - v_{i-\frac{1}{2}}| - |u_{i+\frac{1}{2}} - v_{i+\frac{1}{2}}| \leq |v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}|.
$$

(102)

And,

$$
|\bar{s}_i - s_i| = |F^ε(x_i)| \left| \frac{(f(v_{i+\frac{1}{2}}) - g(v_{i+\frac{1}{2}})) - (f(v_{i-\frac{1}{2}}) - g(v_{i-\frac{1}{2}}))}{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}} \right| \leq \|F^ε\|_{L^∞} \|f - g\|_{Lip}.
$$

Therefore,

$$
B_2 \leq \|F^ε\|_{L^∞} \|f - g\|_{Lip} \sum |v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}| \leq \|F^ε\|_{L^∞} \|f - g\|_{Lip} TV\{v^ε\}.
$$

(103)

Finally, consider the term $B_3$. We have

$$
|\bar{x}_i - \bar{s}_i| = \frac{g(v_{i+\frac{1}{2}}) - g(v_{i-\frac{1}{2}})}{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}} |F^ε(x_i) - G^ε(x_i)|.
$$

Combining this with (102), we get

$$
B_3 \leq \sum |g(v_{i+\frac{1}{2}}) - g(v_{i-\frac{1}{2}})| \cdot |F^ε(x_i) - G^ε(x_i)| \leq \|F^ε - G^ε\|_{L^∞} g'(κ_0) TV\{v^ε\}.
$$

(104)

Concluding the estimate. Putting the estimates (88), (89), (98), (101), (103) and (104) back into (79), we get

$$
\frac{d}{dt} \|u^ε - v^ε\|_{L^1} \leq \|F^ε\|_{L^∞} \|f - g\|_{Lip} TV\{v^ε\} + \|F^ε - G^ε\|_{L^∞} \|f - g\|_{L∞} TV\{F^ε - G^ε\} + C_ε.
$$

(105)

By symmetry we also have

$$
\frac{d}{dt} \|u^ε - v^ε\|_{L^1} \leq \|G^ε\|_{L^∞} \|f - g\|_{Lip} TV\{u^ε\} + \|F^ε - G^ε\|_{L^∞} f'(κ_0) TV\{u^ε\} + TV\{G^ε\} \|f - g\|_{L∞} TV\{F^ε - G^ε\} + C_ε.
$$

(106)

Now let’s estimate the terms in (105)-(106). We have

$$
\|f(v^ε) - g(v^ε)\|_{L^1} = \|[f(v^ε) - f(1)] - [g(v^ε) - g(1)]\|_{L^1},
$$

(107)

Note that it is important to have $f(1) = g(1) = 0$ to obtain (107). For $\|F^ε - G^ε\|_{L^∞}$ we have

$$
\|F^ε - G^ε\|_{L^∞} \leq \max\{\|F^ε\|_{L^∞}, \|G^ε\|_{L^∞}\} \int_{-∞}^{∞} |f(u^ε) - g(v^ε)| \ dy
$$

$$
\leq \max\{\|F^ε\|_{L^∞}, \|G^ε\|_{L^∞}\} \int_{-∞}^{∞} |f(u^ε) - f(v^ε)| + |f(v^ε) - g(v^ε)| \ dy
$$

$$
\leq \max\{\|F^ε\|_{L^∞}, \|G^ε\|_{L^∞}\} [f'(κ_0) \|u^ε - v^ε\|_{L^1} + \|f(v^ε) - g(v^ε)\|_{L^1}]
$$

(108)
and for $TV\{F^\varepsilon - G^\varepsilon\}$ we have
\[
TV\{F^\varepsilon - G^\varepsilon\} = \|F_x^\varepsilon - G_x^\varepsilon\|_{L^1} = \int |f(u^\varepsilon(t,x))F^\varepsilon - g(v^\varepsilon(t,x))G^\varepsilon| \, dx
\]
\[
\leq \int |f(u^\varepsilon) - g(v^\varepsilon)| F^\varepsilon + |g(v^\varepsilon)| |F^\varepsilon - G^\varepsilon| \, dx
\]
\[
\leq \|F^\varepsilon\|_{L^\infty} [f'(\kappa_0) \|u^\varepsilon - v^\varepsilon\|_{L^1} + \|f(v^\varepsilon) - g(v^\varepsilon)\|_{L^1}]
\]
\[
+ \|g(v^\varepsilon)\|_{L^1} \|F^\varepsilon - G^\varepsilon\|_{L^\infty}.
\] (109)

By using (107), (108) and (109), the estimates (105) and (106) become
\[
\frac{d}{dt} \|u^\varepsilon(t,\cdot) - v^\varepsilon(t,\cdot)\|_{L^1}
\]
\[
\leq C \left[ \|u^\varepsilon(t,\cdot) - v^\varepsilon(t,\cdot)\|_{L^1} + \|f - g\|_{L^\infty} + \|f - g\|_{Lip} + \varepsilon \right],
\] (110)
for some bounded constant $C$ that depends only on $\kappa_0$, $M$ and $m_0$. Integrating (110) from 0 to $t$, we obtain the integral estimate (75), completing the proof.

**Remark 7.** The estimates (105) and (106) are very similar to the ones in [17], Theorem 1.3, where the authors study a scalar conservation law with variable coefficients in multi space dimension
\[
u_t + \nabla \cdot (k(x)f(u)) = \Delta A(u).
\]
Continuous dependence on initial data, on the coefficient $k$ and on the flux function $f$ is established with very similar results, by using Kruzkov inequality and a variable doubling technique. However, their coefficient $k(x)$ is local and does not depend on $t$.

On the other hand, the front tracking algorithm proposed here can be easily extended to conservation laws with variable coefficient in one space dimension
\[
u_t + (k(t,x)f(u))_x = 0,
\]
for $k$ under suitable assumptions, such as in [3], Theorem 2. Existence and continuous dependence on initial data, on the coefficient $k$ and on the function $f$ would follow in a similar way.

**Acknowledgements.** The authors wish to thank the referees for their useful comments, which lead to the improvement of the manuscript.

**REFERENCES**


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