MATH 597C, Spring 2007, Home work 2

Problem 1. Consider the traffic flow model:

$$\rho_t + (\rho v)_x = 0,$$

where ρ is the density of the cars, and v is the velocity. Assume that v is a function of ρ

$$v(\rho) = 1 - \rho.$$

Note this is the normalized version as the one we had in class with $\rho_{\max} = 1$ and $v_{\max} = 1$. Construct some smooth initial data $\rho_o(x)$ which will lead to discontinuities in the solution. Try to identify what condition in the initial data would lead to shocks.

Problem 2. Consider the two-phase flow model

$$u_t + f(u)_x = 0, \qquad f(u) = \frac{u^2}{u^2 + a(1-u)^2}, \qquad a = \frac{1}{2}.$$

Solve the Riemann problem with $u_L = 0$ and $u_R = 1$. You may draw the characteristics and sketch the solution at a chosen time, say $t = 1$.

Problem 3. Solve the Burgers' equation $u_t + uu_x = 0$ with the hat-shaped initial data

$$u_0(x) = \begin{cases} 0, & x < -1 \\ x+1, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

Explain in your best possible way what happens in the solution. You may draw the characteristics, mark the shock formation, draw the shock curve etc.

What is the solution $u(x,t)$ as $t \to \infty$?

Problem 4. Solve the "linearized shallow water equations"

$$\begin{aligned} u_t + \bar{u}u_x + v_x &= 0 \\ v_t + \bar{v}u_x + \bar{u}u_x &= 0 \end{aligned}$$

where $\bar{u}$ and $\bar{v} > 0$ are constants, with some suitable initial data

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x).$$