MATH 597C, Spring 2007, Home work 1

Problem 1. Verify that

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases} \qquad \text{where} \qquad s = (u + l + u_r)/2$$

is a weak solution to the Burgers' equation

$$u_t + uu_x = 0, \qquad u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} .$$

Problem 2. Show that the viscous Burgers' equation

$$u_t + uu_x = \varepsilon u_{xx}$$

has a travelling wave solution of the form $u^\varepsilon(x,t) = w(x - st)$ by deriving an ODE for w and verifying that this ODE has solutions of the form

$$w(y) = u_r + \frac{1}{2}(u_l - u_r)[1 - \tanh((u_l - u_r)y/(4\varepsilon))]$$

with $s = (u_l + u_r)/2$. Note that $w(y) \to u_l$ as $y \to -\infty$ and $w(y) \to u_r$ as $y \to +\infty$. Use for example Matlab to plot this solution and observe how it varies as $\varepsilon \to 0$.

Problem 3. There are infinitely many weak solutions to the Riemann problem of Burgers' equation

$$u_t + uu_x = 0, \qquad u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

when $u_l < u_r$. Show, for example, that

$$u(x,t) = \begin{cases} u_l & x < s_m t \\ u_m & s_m t \le x \le u_m t \\ x/t & u_m t \le x \le u_r t \\ u_r & x > u_r t \end{cases}$$

is a weak solution for any u_m with $u_l \le u_m \le u_r$ and $s_m = (u_l + u_m)/2$. Sketch the characteristics for this solution.

Problem 4. Consider the Riemann problem

$$u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases},$$

where $f'' > 0$ (convex) and $u_l < u_r$. Show that the rarefection wave solution is given by

$$u(x,t) = \begin{cases} u_l & x < f'(u_l)t \\ v(x/t) & f'(u_l)t \le x \le f'(u_r)t \\ u_r & x > f'(u_r)t \end{cases}$$

where $v(\xi)$ is the solution to $f'(v(\xi)) = \xi$.