Homework 5; MATH 524: Spring 2008

Due:Fri 04-18-2008

(1). Consider the ODE

$$y' = f(t, y), \quad y(0) = y_0, \quad t \in [0, T],$$

where $f(t, y)$ is Lipschitz in the second variable and continuous in the first. Prove that the implicit Euler method

$$y_{n+1} = y_n + hf(t_n, y_{n+1}), \quad t_n = nh, \quad n = 0, 1, \ldots, N, \quad h = T/N,$$

is convergent, i.e. for any $n \leq N$, one has

$$|y_n - y(t_n)| \to 0, \quad \text{as} \quad h \to 0.$$

(2). Consider the differential equation

$$-\frac{1}{10}u'' + u' = f(x) \quad x \in (0, 1), \qquad u(0) = u(1) = 0\,,$$

and the following finite difference discretization:

$$-\frac{u_{i-1} + u_{i+1} - 2u_i}{10h^2} + \frac{u_{i+1} - u_{i-1}}{2h} = f(x_i), \quad i = 1, \ldots, n,$$

with $u_0 = u_{n+1} = 0$ and mesh size $h = 1/n$. Prove that for sufficiently small h, the discrete problem has unique solution.

(3). Consider the 5-point discretization of the Laplace operator on the 2D domain $\Omega = [0, 1] \times [0, 1]$, with uniform mesh size $h = 1/N$ in both x and y.

a). Show that

$$\phi_{mn}(x, y) = \phi_m(x)\phi_n(y), \quad m, n = 1, \cdots, N - 1$$

where

$$\phi_n(x) = \sin(\pi n x), \quad n = 1, \cdots, N - 1$$

are eigenfunctions for the discrete Laplace operator, with corresponding eigenvalues

$$\lambda_{mn} = \lambda_m + \lambda_n \qquad \text{with } \lambda_n = \frac{2}{h^2}(1 - \cos \pi n h).$$

b). Give an estimate to the condition number of the discrete operator, i.e., $\kappa = \lambda_{\max}/\lambda_{\min}$.

c). If u solves the Poisson equation $\Delta u = f$ in Ω, show the stability estimate

$$\|u\|_h \leq C\|f\|_h$$

where $\|\cdot\|_h$ is the discrete l_2 norm. Find a suitable estimate for the constant C.

(4). Consider one dimensional problem

$$-u'' + u' - u = f, \qquad x \in (0,1),$$

with boundary conditions

$$u(0) = 0, \qquad u'(1) = 0\,.$$

a). Write down the variational formulation of the above differential problem: Find $u \in H_0^1(0,1)$ such that

$$B(u,v) = f(v), \quad \text{for all } v \in H_0^1(0,1).$$

Prove Poincare type inequality in $H_0^1(0,1)$ and use it to show that the above variational problem has a unique solution $u \in H_0^1(0,1)$ for any right hand side $f \in L^2(0,1)$.

b). Let V_h be a finite dimensional subspace of $H_0^1(0,1)$. Show that the discrete problem: Find $u_h \in V_h$ such that

$$B(u_h, v_h) = f(v_h), \quad \text{for all } v_h \in V_h,$$

is well posed and that the following quasi-optimal error estimate holds:

$$\|u - u_h\|_{H_0^1(0,1)} \leq C \inf_{v \in V_h} \|u - v\|_{H_0^1(0,1)}.$$

c). Describe and derive the finite element discretization of this problem with piece-wise linear finite elements. Write down at least one row of the stiffness matrix obtained from this discretization.

d). Prove L_2 and H^1 error estimates for this finite element discretization.