

Qualifying Exam in Numerical Analysis

August 19, 2002

There are ten problems. Six problems fully and correctly solved will guarantee a pass.

- (1) For a given vector b and a given nonsingular upper triangular matrix A , verify that the Jacobi iteration for solving $Ax = b$ is always convergent.

- (2) Let $u, v \in \mathbf{R}^m$ and let $\sigma \in \mathbf{R}^1$. Define

$$H(u, v, \sigma) = I - \sigma uv^T,$$

where I represents the $m \times m$ identity matrix.

a. Determine $\sigma \neq 0$ such that $H(u, u, \sigma)$ is orthogonal. For such σ , determine all the eigenvalues and the corresponding eigenvectors of $H(u, u, \sigma) = I - \sigma uu^T$.

b. Let $x \in \mathbf{R}^m$, $x \neq 0$. Show how to choose a vector u such that $H = H(u, u, \sigma)$ has the property that Hx is a multiple of $e^{(1)} = (1, 0, 0, \dots, 0)^T$ where σ is defined in (a).

c. Show that one can construct orthogonal transformations $H^{(k)}$, $k = 1, \dots, \ell$ such that

$$A^{(\ell+1)} = H^{(\ell)} H^{(\ell-1)} \dots H^{(1)} A, \ell \leq \min(m-1, n)$$

has row echelon structure.

- (3) Given an arbitrary real matrix A ,

a. Describe the singular value decomposition for A .

b. If A is a Hermitian matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, what are the singular values of A ? Justify your answer.

c. Describe how the singular value decomposition can be used to compute the rank of A .

- (4) Given the following recurrence relation on $[-1, 1]$:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \forall n \geq 1.$$

a. Find the general solution.

b. Show that if $T_0(x) = 1$, $T_1(x) = x$, then $T_n(x) = \cos(n \arccos x)$ (the n th Chebyshev polynomial).

c. Prove that $T_{nm}(x) = T_n(T_m(x))$ for all integers $n, m > 0$.

- (5) Consider the function $f(x) = \cos x + 2x$ on the interval $[a, b]$.

a. If $[a, b] = [-\pi/2, \pi/2]$, what is the best uniform approximation to f among all polynomials of degree at most 1?

b. If $[a, b] = [-\pi/2, \pi/2]$, is there a best uniform approximation to f among all functions of the form $c_1 \sin x + c_2 x^2$ where c_1, c_2 are any two real constants? If so, is such a best approximation unique?

(6) Let $h > 0$ be a small parameter

a. Find w_1, w_2 and x_1, x_2 such that the weighted quadrature rule

$$\int_{-h}^h f(x) dx \doteq w_1 f(x_1) + w_2 f(x_2)$$

is exact when f is any cubic polynomial.

b. Give an error formula for the quadrature rule in a) in terms of the derivatives of f and the power of h .

(7) Consider the nonlinear equation $F(x) = 0$, where $F : \Omega \mapsto \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a C^1 function.

a. Derive the Newton's method, namely for a given initial guess x_0 derive the formula for x_{k+1} in terms of x_k if Newton's method is used for the approximate solution of $F(x) = 0$.

b. Assume that $F \in C^3$ and $F'(x_*)$ is non-singular, where x_* is a solution of $F(x) = 0$. Prove that the Newton's method is well defined if x_0 is sufficiently close to x_* and that the sequence of Newton iterates converges quadratically to the solution.

(8) Consider the Runge-Kutta method for $y' = f(x, y)$ with f being smooth:

$$y_{n+1} = y_n + \alpha h f(x_n, y_n) + \frac{h}{2} f(x_n + \beta h, y_n + \beta h f(x_n, y_n)), \quad n = 0, 1, \dots$$

a. For what values of $\{\alpha, \beta\}$, the method is consistent?

b. For what values of $\{\alpha, \beta\}$, the method is stable?

c. For what values of $\{\alpha, \beta\}$, the method is most accurate?

(9) Consider the following differential equation:

$$\begin{cases} -u''(x) + u'(x) - u(x) = f(x), & x \in (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

a. Write down the variational formulation of the above differential problem: Find $u \in H_0^1(0, 1)$ such that

$$B(u, v) = f(v), \quad \text{for all } v \in H_0^1(0, 1).$$

Prove a Poincaré type inequality in $H_0^1(0, 1)$ and use it to show that the above variational problem has a unique solution $u \in H_0^1(0, 1)$ for any right hand side $f \in L^2(0, 1)$.

b. Let V_h be a finite dimensional subspace of $H_0^1(0, 1)$. Show that the discrete problem: Find $u_h \in V_h$ such that

$$B(u_h, v_h) = f(v_h), \quad \text{for all } v_h \in V_h,$$

is well posed and that the following quasi-optimal error estimate holds:

$$|u - u_h|_{H_0^1(0,1)} \leq C \inf_{\chi \in V_h} |u - \chi|_{H_0^1(0,1)}.$$

- (10) Given a constant $a > 0$ and the parabolic partial differential equation $u_t - au_{xx} = 0$ for $x \in (0, 1)$, and $t \in [0, \infty)$ with initial condition $u(x, 0) = u^0(x)$, and periodic boundary condition in space variable x , consider its finite difference discretization on a uniform mesh with steps $h = 1/(N - 1)$ in space and $\tau > 0$ in time:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + a \frac{2u_i^n - u_{i-1}^n - u_{i+1}^n}{h^2} = 0, \quad 2 \leq i \leq N - 1,$$

$$u_i^0 = u_0(ih),$$

$$u_1^{n+1} = u_{N-1}^{n+1},$$

$$u_N^{n+1} = u_2^{n+1}.$$

- a) Find a constant $c > 0$ such that if $\tau \leq ch^2$, then

$$\sum_i |u_i^{n+1}|^2 \leq \sum_i |u_i^n|^2.$$

- b) Verify that under the condition in a), we also have

$$\max_i u_i^{n+1} \leq \max_i u_i^n.$$