QUALIFYING EXAM IN NUMERICAL ANALYSIS

05-11-2005

There are **10 problems** in total in **two sections**. To pass the exam you have to solve **5** out of the **ten** problems below and you should solve at least **two** from each section. Partial credit will be given in borderline cases.

1. Section A

1. Let $a \le x_0, < x_1 \ldots < x_n \le b$ denote $n+1$ distinct points in an interval $[a,b]$. Prove that if $f \in C^{2n+2}[a,b]$ and $p \in \mathcal{P}_{2n+1}$ is its Hermite interpolating polynomial of degree less than or equal to $2n+1$, satisfying $p(x_i) = f(x_i)$, $p'(x_i) = f'(x_i)$, $i = 0, \ldots, n$, then there exists $\xi_x \in (a,b)$, such that for all $x \in (a,b)$ the following relation holds

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!}[\omega(x)]^2, \qquad \text{where}$$

$$\omega(x) = (x-x_0)(x-x_1)\ldots(x-x_n).$$

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2. Consider the linear system $Ax = b$ where A is a real, non-singular 2×2 matrix with nonzero diagonal elements. Show that for such matrix, the Jacobi iterative method converges if and only if the Gauss-Seidel iterations converge.

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3. Let $F : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$ be three times continuously differentiable function on a domain Ω. Assume that the equation $F(x) = 0$ has a unique root $x_* \in \Omega$ and $F'(x)$ is nonsingular for all $x \in \Omega$.

a. Write down the Newton's iteration for the approximate solution of $F(x) = 0$ starting with a given $x_0 \in \Omega$.

b. Show that if x_0 is sufficiently close to x_* then the sequence of Newton's method iterates is well defined and converges quadratically to the root x_*.

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4. Best approximation in Banach space:

a. Let M be a finite dimensional subspace of a real Banach space X. Prove that for any $f \in X$ there exists an element $p \in M$ that minimizes $\|f - q\|_X$ for $q \in M$ (i.e. there exists element of best approximation to f from M).
Hint: First prove that: if $f_0, \ldots, f_n$ are $n+1$ elements of X then the function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ given by $\phi(\mathbf{w}) = \|f_0 - \sum_{i=1}^{n} \mathbf{w}_i f_i\|_X$ is continuous.

b. Show that the set of best approximations to $f \in X$ with elements from M is convex.

c. Assuming that X and M are such that the best approximation is unique, is the operator mapping f to its best approximation $P_M(f) \in M$ always linear?

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5. Best approximation in Hilbert space:

a. Let M be a closed subspace of a real Hilbert space H. Prove that for any $u \in H$ there exists unique element $u_M \in M$ that minimizes $\|u - v_M\|_H$ for $v_M \in M$.

b. Let $b(\cdot,\cdot)$ be a continuous, symmetric and positive definite bilinear form $b : H \times H \mapsto \mathbb{R}$, namely for all u and v from H, the following are satisfied for $\gamma > 0$ and $\beta > 0$.

$$b(u,u) \geq \gamma \|u\|_H^2; \qquad |b(u,v)| \leq \beta \|u\|_H \|v\|_H.$$

Prove that if $\hat{u}_M$ satisfies $b(u - \hat{u}_M, v_M) = 0$, for all $v_M \in M$, then

$$\|u - \hat{u}_M\|_H \leq \frac{\beta}{\gamma} \|u - u_M\|_H,$$

where u_M (as in **a.**) is the element of best approximation to u from M.

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2. Section B

1. Assume that the solution $y(t)$ to the initial value problem

$$y' = f(t,y); \qquad t \in [0,T], y(0) = y_0,$$

is twice continuously differentiable, i.e. $y \in C^2([0,T])$ and that f has continuous first derivatives with respect to x and t. Prove that prove that

$$\lim_{h \to 0^+} \max_{0 \leq n \leq N} |y(t_n) - y_n| = 0,$$

where the approximations y_n are obtained via the explicit Euler method $y_{n+1} = y_n + hf(t_n, y_n)$. Here $t_n = nh$, $0 \leq n \leq N$ and $h = T/(N-1)$, $N > 1$.

Hint: You may use the following fact, without proving it: If $A, B, \eta_0, \eta_1, ..., \eta_N$ are non-negative numbers satisfying $\eta_{n+1} \leq A\eta_n + B, n = 0, 1, ..., N-1$ then $\eta_n \leq A^n \eta_0 + B \sum_{i=0}^{n-1} A^i, \quad n = 0, 1, ..., N.$

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2. Consider a general n-point numerical quadrature for computing the integral of f with weight $w(x)$, $w(x) > 0$ on $[-1,1]$:

$$\int_{-1}^{1} f(x)w(x)\, dx \approx \sum_{i=1}^{n} \omega_i f(x_i),$$

and let $\mathcal{P}_k$ denote the linear space of polynomials of degree less than or equal to k.

a. Let $p_n \in \mathcal{P}_n$ be such that $\int_{-1}^{1} p_n(x) q_{n-1}(x) w(x)\, dx = 0, \quad$ for all $\quad q \in \mathcal{P}_{n-1}$. Assume that $\{x_i\}_{i=1}^n$ are the roots of this polynomial (you may use without a proof that the roots of such a polynomial are all real and distinct). Prove that the weights ω_i can be determined in a way that the quadrature is exact for all $p \in \mathcal{P}_{2n-1}$.

b. Show that there can not be a quadrature rule of the form described above, that is exact for all polynomials of degree $2n$.

c. Prove that the weights $\{\omega_i\}_1^n$ determined in **a.** are positive.

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3. Consider the conjugate gradient method for the minimization of $\frac{1}{2}(Au, u) - (b, u)$. ($A$ is a symmetric and positive definite matrix) in the form: Starting with $u^0 = 0$, $r_0 = b$ and $p_0 = r_0$ the successive approximations to the minimizer are computed by

$$\begin{aligned} u^{k+1} &= u^k + \alpha_k p_k, \qquad r_{k+1} = r_k - \alpha_k A p_k; \\ p_{k+1} &= r_{k+1} - \beta_k p_k \end{aligned}$$

where $\alpha_k = (r_k, p_k)/\|p_k\|_A^2$ and $\beta_k = -(r_{k+1}, p_k)_A/\|p_k\|_A^2$.

a. Show that for $k = 0, 1, 2, ...$ the following relations are true

$$span\{p_0, p_1, ..., p_k\} = span\{r_0, r_1, ..., r_k\} = span\{r_0, Ar_0, ..., A^k r_0\}.$$

b. Show that if $A \in R^{n \times n}$ then for some $m \le n$, $r_m = 0$ (assume that all operations are performed exactly).

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4. Consider the boundary value problem on $\Omega = (0,1) \times (0,1)$,

$$-\Delta u = 0, \quad \text{in} \quad \Omega; \qquad u = x + 2 > 0 \quad \text{on} \quad \partial\Omega,$$

discretized by the usual 5 point difference scheme:

$$\Delta_h U \equiv \frac{4U_{j,k} - U_{j+1,k} - U_{j-1,k} - U_{j,k+1} - U_{j,k-1}}{h^2} = 0.$$

on an equidistant grid Ω_h with a step size h.

a. Suppose that the grid (or mesh) function U satisfies the boundary condition exactly. Can the solution of this difference scheme, U, achieve its maximum inside $\Omega_h = \{(x_i, y_j), \quad 0 < x_i < 1, \quad 0 < y_j < 1\}$? Justify your answer.

b. Suppose that the exact solution u of the boundary value problem is four times continuously differentiable, i.e. $u \in C^4(\bar{\Omega})$ and that the solution of the difference scheme U satisfies the boundary conditions exactly, prove that there exist a positive constant c such that

$$\|u - U\|_{\infty,\Omega_h} := \max_{i,j} |u_{i,j} - U_{i,j}| \le ch^2,$$

where $u_{i,j} = u(ih, jh)$.

Hint: You may use the following fact without proving it: If the grid (mesh) function V satisfies homogeneous boundary conditions then, for this particular domain: $\|V\|_{\infty,\Omega_h} \le \frac{1}{8}\|\Delta_h V\|_{\infty,\Omega_h}$.

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5. Use von Neumann analysis to show the stability of the following finite difference scheme for the advection equation with periodic conditions

$$u_t + cu_x = 0; \qquad u(x,0) = u_0(x); \qquad u(x+1,t) = u(x,t),$$

discretized with central finite difference scheme in space, that is

$$U_n^{j+1} + \frac{\lambda}{4}(U_{n+1}^{j+1} - U_{n-1}^{j+1}) = U_n^j - \frac{\lambda}{4}(U_{n+1}^j - U_{n-1}^j),$$

where $\lambda = ck/h$ and U_n^j approximates the value of the solution $u(nh, jk)$, where k is the time step and h is the dicretization step in space.