

QUALIFYING EXAM IN NUMERICAL ANALYSIS

08-22-2005

There are **10 problems** in total in **two sections**. To pass the exam you have to solve **5** out of the **ten** problems below and you should solve at least **two** from each section. Partial credit will be given in borderline cases.

1. SECTION A

1. Consider the midpoint quadrature rule on the unit interval

$$\int_0^1 f(x) dx \approx f\left(\frac{1}{2}\right),$$

where $f(\cdot)$ is a smooth function.

- a. Find the Peano kernel representation of the error when the integral is approximated by the above rule.
 - b. Write down the composite midpoint rule on a given finite interval $[a, b]$, with step size $h = (b - a)/N$, ($N > 1$ is an integer). Using the result from the previous item, prove an error bound (in terms of h , a , b and derivatives of $f(\cdot)$) for the composite rule.
2. • State Jackson's theorem for the approximation of C^1 periodic functions by trigonometric polynomials.
• State Jackson's theorem for the approximation of C^1 functions on the unit interval by ordinary polynomials.
• Prove the latter using the former.
3. Consider the reversed Lax-Friedrichs scheme:

$$\frac{U_{n+1}^{j+1} - 2U_n^j + U_{n-1}^{j+1}}{2k} + c \frac{U_{n+1}^{j+1} - U_{n-1}^{j+1}}{2h} = 0.$$

for the advection equation ($c > 0$) with periodic conditions (for $t > 0$):

$$u_t + cu_x = 0; \quad u(x, 0) = u_0(x); \quad u(x + 1, t) = u(x, t).$$

Here U_j^n approximates the value of the solution $u(jk, nh)$, where k is the time step and h is the discretization step in space.

- a. Show that if $\lambda = ck/h$ then this scheme can be written as:

$$(1 + \lambda)U_{n+1}^{j+1} + (1 - \lambda)U_{n-1}^{j+1} = 2U_n^j.$$

- b. Use von Neumann analysis to investigate the stability of this scheme.
c. Show that this scheme can be viewed as a discretization of the following differential operator:

$$u_t + cu_x + \frac{h^2}{2k}u_{xx}.$$

4. Gaussian elimination and LU factorization:

- a. Let A be a symmetric and positive definite matrix of order n , $A = \begin{pmatrix} a_{11} & \mathbf{a}^T \\ \mathbf{a} & A_{n-1} \end{pmatrix}$, where \mathbf{a} is a column vector and A_{n-1} is an $(n-1) \times (n-1)$ matrix. After one step of Gaussian elimination A is converted to a matrix of the form $\begin{pmatrix} a_{11} & \mathbf{a}^T \\ 0 & \tilde{A} \end{pmatrix}$.

Show that the $(n-1) \times (n-1)$ matrix $\tilde{A} := A_{n-1} - (\mathbf{a}\mathbf{a}^T)/a_{11}$ is symmetric and positive definite.

- b. Let (p_1, p_2, \dots, p_n) be a permutation of $(1, 2, \dots, n)$. Define the permutation matrix P as

$$P_{ij} = \begin{cases} 1, & i = p_j \\ 0, & i \neq p_j \end{cases}, \quad i = 1, 2, \dots, n.$$

Prove or disprove: if A is nonsingular (not necessarily symmetric and positive definite), then there exists a permutation matrix P , and a pair: *unit* lower triangular matrix L and upper triangular matrix U such that $PA = LU$.

- c. Show that if all principal minors of A are nonsingular, P can be taken to be the identity and in such case the pair L, U is unique.
5. Find the quadratic polynomial which is the best approximation to the function $f(x) = x^3$ with respect to the maximum norm on the interval $[0, 1]$. Justify your answer.

2. SECTION B

1. Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary $\partial\Omega$. Consider the following boundary value problem:

$$\begin{cases} -\Delta u + u_x = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

- a. Write down the variational formulation of the above differential problem: Find $u \in H_0^1(\Omega)$ such that

$$B(u, v) = f(v), \quad \text{for all } v \in H_0^1(\Omega).$$

Show that this variational problem has a unique solution $u \in H_0^1(\Omega)$ for any right hand side $f \in L^2(\Omega)$.

- b. Let V_h be a finite dimensional subspace of $H_0^1(\Omega)$. Show that the discrete problem: Find $u_h \in V_h$ such that

$$B(u_h, v_h) = f(v_h), \quad \text{for all } v_h \in V_h,$$

is well posed and that the following quasi-optimal error estimate holds:

$$|u - u_h|_{H_0^1(\Omega)} \leq C \inf_{\chi \in V_h} |u - \chi|_{H_0^1(\Omega)}.$$

2. Consider the conjugate gradient method for the minimization of $\frac{1}{2}(Au, u) - (b, u)$. (A is a symmetric and positive definite matrix) in the form: Starting with $u^0 = 0$, $r_0 = b$ and $p_0 = r_0$ the successive approximations to the minimizer are computed by

$$\begin{aligned} u^{k+1} &= u^k + \alpha_k p_k, & r_{k+1} &= r_k - \alpha_k A p_k; \\ p_{k+1} &= r_{k+1} - \beta_k p_k \end{aligned}$$

where $\alpha_k = (r_k, p_k) / \|p_k\|_A^2$ and $\beta_k = -(r_{k+1}, p_k)_A / \|p_k\|_A^2$.

- a. Show that for $k = 0, 1, 2, \dots$ the following relations are true

$$\text{span}\{p_0, p_1, \dots, p_k\} = \text{span}\{r_0, r_1, \dots, r_k\} = \text{span}\{r_0, Ar_0, \dots, A^k r_0\}.$$

- b. Show that if $A \in R^{n \times n}$ then for some $m \leq n$, $r_m = 0$ (assume that all operations are performed exactly).

3. Let A be a diagonally dominant M -matrix. Prove that the Jacobi and Gauss-Seidel iterative methods both converge to the solution of $Ax = b$, for any initial guess.
4. Consider the implicit scheme

$$(1) \quad y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1})).$$

for the solution of the initial value problem:

$$(2) \quad y' = f(t, y), \quad y(0) = y_0, \quad t \in [0, T].$$

- a. Assume that the right hand side f is Lipschitz continuous, and show that for sufficiently small step size h , (??) has a solution y_{n+1} for any y_n .
- b. Assume that the solution $y(t)$ to the initial value problem (??) is twice continuously differentiable, i.e. $y \in C^2([0, T])$, and prove that

$$\lim_{h \rightarrow 0} |y(t_n) - y_n| = 0,$$

where the approximations y_n are obtained via the implicit trapezoid scheme. Here $t_n = nh$, $0 \leq n \leq N$ and $h = T/N - 1$, $N > 1$.

5. Let \hat{T} and T be the following non-degenerate simplexes,

$$\hat{T} := \{\hat{\mathbf{x}} \in \mathbb{R}^2 \mid \hat{x}_i \geq 0, \quad i = 1, 2; \quad \hat{x}_1 + \hat{x}_2 \leq 1\},$$

$$T := \{\mathbf{x} \in \mathbb{R}^2, \mid \mathbf{x} = B\hat{\mathbf{x}} + \mathbf{x}_0, \quad \hat{\mathbf{x}} \in \hat{T}\},$$

where B is a nonsingular 2×2 matrix and $\mathbf{x}_0 \in \mathbb{R}^2$ is a fixed vector.

- a. Find the three linear functions, $\{\hat{\phi}_i(\hat{\mathbf{x}})\}_{i=0}^2$, such that $B\hat{\mathbf{x}} + \mathbf{x}_0 = \sum_{i=0}^2 \hat{\phi}_i(\hat{\mathbf{x}})\mathbf{x}_i$,

where \mathbf{x}_i are the vertices of T .

- b. Assume that $f \in H^2(T)$. Give an estimate of $\|f - f_I\|_{L_2}$ and of $\|\nabla(f - f_I)\|_{L_2}$. Justify your answer. Here f_I is the linear interpolant of $f(\mathbf{x})$, defined as

$$f_I(\mathbf{x}) := \sum_{i=0}^2 f(\mathbf{x}_i)\phi_i(\mathbf{x}), \quad \phi_i(\mathbf{x}) = \hat{\phi}_i(B^{-1}(\mathbf{x} - \mathbf{x}_0)).$$