Qualifying Exam in Numerical Analysis
August 17, 2001

There are ten problems. Six problems fully and correctly solved will guarantee a pass.

(1) Let \( T_n(x) \) denote the \( n \)th Chebyshev polynomial on the interval \([-1, 1]\) defined by using the following recurrence relation
\[
T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).
\]

a. Show that \( T_n(x) = \cos(n \arccos x) \).

b. Prove that
\[
\int_{-1}^{1} T_n T_m \frac{1}{\sqrt{1-x^2}} \, dx = 0 \quad \text{for all integers } n \text{ and } m, \text{ such that } n \neq m, \, n, m > 0.
\]

c. Prove that \( T_{nm}(x) = T_n(T_m(x)) \) for all integers \( n, m > 0 \).

a. Let \( \alpha = \arccos x \). It is sufficient to show that \( \cos(n \arccos x) \) satisfies the same recurrence relation as \( T_n \). For \( n = 0, 1 \) the result is trivial. Then a. follows from the fact that
\[
\cos(n + 1)\alpha + \cos(n - 1)\alpha = 2 \cos \alpha \cos n\alpha.
\]

b. and c. are easy and straightforward applications of a.

(2) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \). Consider the following partial differential equation:
\[
\begin{cases}
-\Delta u + u_x &= f, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega.
\end{cases}
\]

a. Write down the variational formulation of the above differential problem: Find \( u \in H^1_0(\Omega) \) such that
\[
 B(u, v) = f(v), \quad \forall v \in H^1_0(\Omega).
\]

Show that this variational problem has a unique solution \( u \in H^1_0(\Omega) \) for any right hand side \( f \in L^2(\Omega) \).

b. Let \( V_h \) be a finite dimensional subspace of \( H^1_0(\Omega) \). Show that the discrete problem: Find \( u_h \in V_h \) such that
\[
 B(u_h, v_h) = f(v_h), \quad \forall v_h \in V_h,
\]
is well posed and that the following quasi-optimal error estimate holds:
\[
|u - u_h|_{H^1_0(\Omega)} \leq C \inf_{\chi \in V_h} |u - \chi|_{H^1_0(\Omega)}.
\]

a. Variational form is: Find \( u \in H^1_0(\Omega) \) such that
\[
 B(u, v) = f(v), \quad \forall v \in H^1_0(\Omega),
\]

where as usual
\[
 B(u, v) = \int_{\Omega} \nabla u \nabla v + u_x v \, dx, \quad f(v) = \int_{\Omega} fv \, dx.
\]

Simple integration by parts leads to
\[
 B(u, u) = |u|^2_{H^1_0(\Omega)}.
\]
and from this equality the Lax-Milgram lemma gives the result from (a.)

b. The discrete problem is well posed by the same token as in a. To prove the bound, let \( \chi \in V_h \) be arbitrary. Note that
\[
B(u - u_h, v_h) = 0, \quad \forall v_h \in V_h,
\]
and also by Schwarz inequality and Poincare inequality
\[
B(u, v) \leq C|u|_{H^1_0(\Omega)}|v|_{H^1_0(\Omega)}.
\]
Combining the above two results we obtain that
\[
|u - u_h|^2_{H^1_0(\Omega)} = B(u - u_h, u - u_h) = B(u - u_h, u - \chi) \leq C|u - u_h|_{H^1_0(\Omega)}|u - \chi|_{H^1_0(\Omega)}.
\]
and the proof of (b.) is completed by taking the infimum over \( \chi \in V_h \).

(3) Consider the nonlinear equation \( F(x) = 0 \), where \( F : \Omega \rightarrow \mathbb{R}^n, \Omega \subset \mathbb{R}^n \) is a \( C^1 \) function.

a. Derive the Newton’s method, namely for a given initial guess \( x_0 \) derive the formula for \( x_{k+1} \) in terms of \( x_k \) if Newton’s method is used for the approximate solution of \( F(x) = 0 \).

b. Assume that \( F \in C^3 \) and \( F'(x_*) \) is non-singular, where \( x_* \) is a solution of \( F(x) = 0 \). Prove that the Newton’s method is well defined if \( x_0 \) is sufficiently close to \( x_* \) and that the sequence of Newton iterates converges quadratically to the solution.

a. By Taylor’s formula we have that
\[
F(x) \approx F(x_0) + [F'(x_0)](x - x_0).
\]
In Newton’s method an approximation to the root is obtained by solving the approximate equation, which is linear with respect to \( x \). So given \( x_k \) we have that the next iterate \( x_{k+1} \) is obtained by
\[
x_{k+1} = x_k - [F'(x_k)]^{-1}F(x_k).
\]
b. Clearly, if \( x_k \) is sufficiently close to \( x_* \), we have that \( F'(x_k) \) is non-singular (because is continuous and non-singular at \( x_* \)). So we have to prove that if \( x_k \) is in a small neighborhood of \( x_* \), then \( x_{k+1} \) will stay in the same neighborhood. Let \( G(x) := x - [F'(x)]^{-1}F(x) \). Clearly \( x_* \) is a fixed point of \( G \). A simple calculation gives
\[
G'(x) = I - K(x)F(x) - [F'(x)]^{-1}F'(x) = -K(x)F(x),
\]
where
\[
K(x) = ([F'(x)]^{-1})' = -[F'(x)]^{-1}[F''(x)][F'(x)]^{-1}.
\]
Note also that for \( x, y \in \mathbb{R}^n \)
\[
G(y) - G(x) - G'(x)(y - x) = \left( \int_0^1 [G''(x + t(y - x)) - G'(x)] dt \right)(y - x)
\]
We have that \( G'(x) \) is Lipschitz (it is even differentiable, because \( F \in C^3 \)) and this gives the following estimate:
\[
\|G(y) - G(x) - G'(x)(y - x)\| \leq \frac{C}{2}\|x - y\|^2,
\]
where \( C \) is the Lipschitz constant (or a bound on the second derivative of \( G \) in case when \( F(x) \in C^3 \)). Taking \( x = x_*, y = x_k \) and using that \( G'(x_*) = 0 \) we obtain
\[
\|x_{k+1} - x_*\| \leq \frac{C}{2}\|x_k - x_*\|^2.
\]
(4) Consider the initial value problem
\[ y' = f(t, y), \quad y(0) = y_0. \]

a. Derive an explicit, two-stage, second order Runge-Kutta method for the approximate solution of this problem of the form
\[ y_{n+1} = y_n + h[\alpha_1 f(t_n, y_n) + \alpha_2 f(t_n + \theta h, y_n + k_n)]. \]

Justify your answer.

Let us set \( k_n = \beta hf(t_n, y_n). \) We compare
\[ y_{n+1} = y_n + h[\alpha_1 f(t_n, y_n) + \alpha_2 f(t_n + \theta h, y_n + k_n)]. \] \hspace{1cm} (1)

with
\[ y_{n+1} = y_n + hf + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \ldots \] \hspace{1cm} (2)

trying to match the coefficients in front of equal powers of \( h. \) Applying Taylor formula for \( f(t_n + \theta h, y_n + k_n) \) then gives
\[ f(t_n + \theta h, y_n + \beta hf) = f + \theta hf_t + \beta h f_y + \mathcal{O}(h^2), \]

where \( f = f(t_n). \) After substitution in (1) we get
\[ y_{n+1} = y_n + (\alpha_1 + \alpha_2) hf + \theta \alpha_2 h^2 f_t + \beta \alpha_2 h^2 f_y + \mathcal{O}(h^3). \]

Note that \( y' = f \) gives \( y'' = f_t + f f_y. \) Therefore 2 takes the form:
\[ y_{n+1} = y_n + hf + \frac{h^2}{2}(f_t + f f_y) + \mathcal{O}(h^3). \]

This leads to the following equations for \( \alpha_i, \beta \) and \( \theta. \)
\[ \alpha_1 + \alpha_2 = 1, \quad \theta \alpha_2 = \frac{1}{2}, \quad \beta \alpha_2 = \frac{1}{2}. \]

There are many solutions to these equations. A popular one is obtained when \( \alpha_1 = \alpha_2 \) and the corresponding method is given below.
\[ y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + k_n)], \quad k_n = hf(t_n, y_n) \]

(5) a. Find \( \alpha \) and \( \beta \) such that the weighted quadrature rule
\[ \int_0^1 \frac{f(x)}{\sqrt{x}} dx = \alpha f(0) + \beta f(1) \]
is exact when \( f \) is linear.

b. Give the Peano kernel error formula for quadrature rule from (a.)

a.
\[ \int_0^1 \frac{f(x)}{\sqrt{x}} dx = \frac{4}{3} f(0) + 2 \frac{2}{3} f(1). \]

b. We note that the above rule is exact if \( f \in \mathcal{P}_1. \) The Peano kernel theorem then gives:
\[ \int_0^1 \frac{f(x)}{\sqrt{x}} dx - \left[ \frac{4}{3} f(0) + 2 \frac{2}{3} f(1) \right] = \int_0^1 f''(t) K(t) dt, \]
where $K(t)$ is the error in approximating $(x - t)_+$ by the above quadrature rule (the integration is done with respect to $x$. This gives the following expression for $K(t)$:

\[
\int_0^1 \frac{(x-t)_+}{\sqrt{x}}\,dx - \frac{2}{3}(1-t) = \frac{4}{3}t(\sqrt{t} - 1).
\]

(6) Let $A$ be the following $2 \times 2$ matrix

\[
A = \begin{pmatrix} a & -b \\ -a & a \end{pmatrix},
\]

where $a$ and $b$ are real numbers, satisfying $a > 0$, $b > 0$ and $a > b$. Show that Gauss-Seidel iteration is convergent for this type of matrices.

The Gauss-Seidel iteration for the matrix $A$ will be convergent iff $\rho(I - BA) < 1$, where

\[
B = \begin{pmatrix} a & 0 \\ -a & a \end{pmatrix}^{-1} = \frac{1}{a} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

This gives

\[
I - BA = \frac{b}{a} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}
\]

From this equation it is straightforward to find that $\rho(I - BA) = \frac{b}{a} < 1$.

(7) Consider the space $\mathcal{P}_2$ of all quadratic polynomials on $[0, 2]$.

a. Prove that the expression

\[
\|f\| := |f(0)| + |f(1)| + |f(2)|, \quad f \in \mathcal{P}_2,
\]

defines a norm on $\mathcal{P}_2$.

b. Determine a best approximation to $f(x) = x^2$ by the constant functions with respect to this norm.

c. Is this best constant approximation to $f(x) = x^2$ unique? Justify your answer.

a. The proof that $\|\cdot\|$ is a norm is straightforward. First observe that $f(0) = f(1) = f(2) = 0$ implies that $f \equiv 0$ for a quadratic polynomial $f$. The other properties easily follow from similar ones for the absolute value. b. Let $p$ be the approximation under question. It follows that $p$ minimizes

\[
g(p) = |p| + |p - 1| + |p - 4|.
\]

Evidently $g$ is a piece-wise linear function and its minimal value is achieved at one of the critical points (where $g'(p)$ does not exists). We then easily find that such a point is $p = 1$ and is unique.
(8) Given the following parabolic partial differential equation
\[ u_t - \Delta u = 0, \quad x \in \Omega = (0, 1) \times (0, 1), \quad t \in [0, \infty) \]
\[ u(x, 0) = u^0(x), \]
\[ u(x, t) = 0, \quad x \in \partial \Omega, \quad t \in [0, \infty), \]
consider its finite difference discretization on a uniform \( N \times N \) mesh with steps \( h = \frac{1}{N-1} \) in space and \( \tau > 0 \) in time:
\[
\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\tau} + \frac{4u^{n+1}_{i,j} - u^n_{i,j} - u_{i+1,j}^{n+1} - u_{i,j-1}^{n+1} - u_{i,j+1}^{n+1} - u_{i,j}^{n+1}}{h^2} = 0, \quad 2 \leq i, j \leq N - 1,
\]
where
\[ u^n_{i,j} = u(x_i, y_j, n\tau), \quad x_i = (i-1)h, \quad y_j = (j-1)h, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N. \]

a. Let \( L_h u \) denotes the stationary part of the above finite difference operator, namely:
\[ L_h u := \frac{4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}}{h^2}, \quad 2 \leq i, j \leq N - 1. \]

Show that \( I + \tau L_h \) satisfies the following maximum principle:

If \((I + \tau L_h)u \geq 0 \text{ and } u_{i,j} \geq 0 \text{ for } (x_i, y_j) \in \partial \Omega, \text{ then } u_{i,j} \geq 0, \quad 1 \leq i, j \leq N\)

where \( I \) denotes the identity operator.

b. Prove that
\[ \max_{i,j} u^{n+1}_{i,j} \leq \max_{i,j} u^n_{i,j}. \]

To prove a. we will show that if \((I + L_h)u \geq 0 \text{ and } u_{i,j} \text{ has a local minimum in an internal point } (x_{i_0}, y_{j_0}), \text{ then } u_{i_0,j_0} \geq 0. \)

Let
\[ u_{i_0,j_0} \leq u_{k,l}, \quad k = i_0 - 1, i_0 + 1; \quad l = j_0 - 1, j_0 + 1. \]

Then
\[
0 \leq (I + \tau L_h)u = u_{i_0,j_0} + \frac{\tau}{h^2} (4u_{i_0,j_0} - u_{i_0-1,j_0} - u_{i_0+1,j_0} - u_{i_0,j_0-1} - u_{i_0,j_0+1}) \leq u_{i_0,j_0}.
\]

The boundary condition gives that the desired inequality is satisfied on the boundary of the domain and the desired result follows.

b. Let \( u^n_{\max} := \max_{i,j} u^n_{i,j} \) and
\[ w^n_{i,j} := u^n_{\max}, \quad \text{for all } i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N. \]

To prove b. it is sufficient to show that \( u^n_{\max} \geq u_{i,j}^{n+1}, \quad \forall \ i, j. \)

Note that \( u^n \) and \( u^{n+1} \) satisfy the relation
\[ (I + \tau L_h)u^{n+1} = u^n, \quad u_{i,j}^{n+1} = 0 \quad (x_i, y_j) \in \partial \Omega. \]

We also have that in the interior of \( \Omega, \)
\[ 0 \leq w^n - u^n = (I + \tau L_h)(w^n - u^{n+1}). \]

Since \((I + \tau L_h)\) satisfies maximum principle complete the proof by applying (a.)
Let \( A \in \mathbb{R}^{m \times n} \) have rank \( n \), and \( b \in \mathbb{R}^m \).

a. Show that the matrix \( A^T A \) is invertible.

b. Show that there exists a unique \( x \in \mathbb{R}^n \) minimizing \( Ax - b \) with respect to the Euclidean norm and \( x = (A^T A)^{-1}A^T b \).

Let us first note that \( n \leq m \) because \( A \) has rank \( n \). Since we consider finite dimensional space we shall prove a. by showing that \( \text{Ker}(A^T A) = \{0\} \) or equivalently that \( A^T A \) is injective. Assume that there is an \( x \in \mathbb{R}^n \) such that \( A^T A x = 0 \). We then have
\[
0 = (A^T A x, x) = \|Ax\|^2 \implies Ax = 0.
\]
But \( A \) has rank \( n \) which exactly means that there is no \( x \neq 0 \) for which \( Ax = 0 \) and therefore \( A^T A \) is injection. This in turn implies \( A^T A \) is invertible.

b. To prove the statement we first observe that \( F(x) = \|Ax - b\|^2 \) is a quadratic functional, namely
\[
F(x) = (A^T A x, x) - 2(A^T b, x) + \|b\|^2.
\]
Note that \( F \) has a unique minimum, because by a. \( A^T A \) is invertible and hence positive definite. The Euler-Lagrange equations for such functional then are:
\[
A^T A x = A^T b.
\]
and so the solution of \( \min_{y \in \mathbb{R}^n} F(y) \) is \( x = (A^T A)^{-1}A^T b \).

Let \( A = (a_{ij}) \) be a symmetric and positive definite matrix of order \( n \).

\[
A = \begin{pmatrix}
a_{11} & a_{12}^T \\
a & A_1
\end{pmatrix}.
\]

After one step of Gaussian elimination \( A \) is converted to a matrix of the form
\[
\begin{pmatrix}
a_{11} & a_{12}^T \\
0 & \tilde{A}
\end{pmatrix}.
\]

a. Find an explicit formula for \( \tilde{A} \) in terms of \( a_{11}, A_1 \) and \( a \).

b. Show that the \((n - 1) \times (n - 1)\) matrix \( \tilde{A} \) is symmetric and positive definite.

a. \( \tilde{A} = A_1 - \frac{aa^T}{a_{11}} \).

b. Let \( x \in \mathbb{R}^{n-1} \) be arbitrary and \( x_1 := -\frac{(a, x)}{a_{11}} \). Consider
\[
y = \begin{pmatrix} x_1 \\ x \end{pmatrix}
\]
Note that \( (Ay, y) = (\tilde{A}x, x) \). Since \( A \) is symmetric positive definite, we have that there exists a number \( \gamma \) such that \( (Ay, y) \geq \gamma \|y\|^2 \). Therefore
\[
\gamma \|x\|^2 \leq \gamma \|y\|^2 \leq (Ay, y) = (\tilde{A}x, x).
\]