

**SOLUTION OF QUALIFYING EXAM IN NUMERICAL  
ANALYSIS 2005**

**Section A.** 1. Let us recall that if  $q$  is the Lagrange interpolant of  $f$  based on points  $x_i$ , then for any  $x \in [a, b]$  there exist  $\xi_x$  such that

$$(1) \quad f(x) - q(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \omega(x).$$

Without loss of generality, we assume  $a \leq x_0 < x_1 < \dots < x_n \leq b$ . For any  $0 < \epsilon < 0.25 \min |x_i - x_{i-1}|$ , we consider the Lagrange interpolant  $p_\epsilon$  at  $2n + 2$  points  $x_0, x_0 + \epsilon, x_1 + \epsilon, \dots, x_n - \epsilon$ . Based on (1), we know

$$f(x) - p_\epsilon(x) = \frac{f^{(2n+2)}(\xi_x^\epsilon)}{(2n+2)!} \omega^\epsilon(x),$$

where  $\omega^\epsilon(x) = (x - x_0)(x - x_0 - \epsilon) \dots (x - x_n)(x - x_n + \epsilon)$ . It is trivial to show that for any  $x \in [a, b]$

$$\lim_{\epsilon \rightarrow 0} p_\epsilon(x) = p(x), \quad \lim_{\epsilon \rightarrow 0} \omega^\epsilon(x) = \omega^2(x),$$

and there exist a subsequence of  $\xi_x^\epsilon$  such that it converges to some point  $\xi_x \in [a, b]$ .

4. a. By triangle inequality

$$|\phi(x) - \phi(y)| \leq \left\| \sum (x_i - y_i) f_i \right\| \leq \sum |x_i - y_i| \|f_i\| \leq \max_i \|f_i\| \sum |x_i - y_i|.$$

Therefore  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous.

Let us restrict this function on the domain

$$S := \{x \in \mathbb{R}^n : \left\| \sum x_i f_i \right\| \leq 2\|f\|\}.$$

It is easy to show  $S$  is compact and thus the continuous function  $\phi$  attains a minimum on  $S$ , say at  $x_0$ . For  $x \in \mathbb{R}^n \setminus S$ , we have

$$\phi(x) \geq \left\| \sum x_i f_i \right\| - \|f\| > \|f\| = \phi(0) \geq \phi(x_0).$$

This shows that  $x_0$  is a global minimizer. b. If the best approximation  $P(f)$  is unique, the set  $\{P(f)\}$  is convex. Otherwise let  $p, q$  be two best approximations and let  $0 < \theta < 1$ , we have

$$\inf_{r \in M} \|f - r\| \leq \|f - [\theta p + (1 - \theta)q]\| \leq \theta \|f - p\| + (1 - \theta) \|f - q\| = \inf_{r \in M} \|f - r\|,$$

which shows that  $\theta p + (1 - \theta)q$  is also a best approximation of  $f$ .

c.  $P(f)$  is not well defined. Even it is unique, it is not linear.

5. a. Let  $(\cdot, \cdot)$  denote the inner product of  $H$ . For any  $u \in H$ ,  $(u, \cdot)$  defines a linear functional on  $M$ . Therefore by Riesz theorem, there exists a unique element  $u_M \in M$  such that

$$(u_M, v) = (u, v), \quad \text{for } v \in M.$$

We are going to prove  $u_M$  is the unique best approximation. Indeed

$$\|u - u_M\|^2 = (u - u_M, u - u_M) = (u - v, u - u_M) \leq \|u - v\| \|u - u_M\|$$

and thus

$$\|u - u_M\| \leq \|u - v\|.$$

The equality holds if and only if  $u - v = u - u_M$  which is equivalent to  $v = u_M$ . This shows the uniqueness.

b.

$$\begin{aligned} \|u - \hat{u}_M\|^2 &= \frac{1}{\gamma} b(u - \hat{u}_M, u - \hat{u}_M) \\ &= \frac{1}{\gamma} b(u - u_M, u - \hat{u}_M) \\ &\leq \frac{\beta}{\gamma} \|u - u_M\| \|u - \hat{u}_M\|. \end{aligned}$$

**Section B.** 2. a. Let  $l_i(x)$  be the Lagrange basis of points  $\{x_i\}$  at point  $x_i$ . We define

$$\omega_i = \int_{-1}^1 l_i(x) w(x) dx.$$

We are going to prove the numerical quadrature using weights defined above is  $2n - 1$  exact. Let us denote

$$I(f) = \int_{-1}^1 f(x) w(x) dx \quad \text{and} \quad J(f) = \sum_i \omega_i f(x_i).$$

For a function  $f \in \mathcal{P}_{2n-1}$ , it can be decomposed as  $f(x) = q(x)p_n(x) + r(x)$ , where  $q(x), r(x) \in \mathcal{P}_{n-1}$ . Note that

$$I(q \cdot p_n) = \int_{-1}^1 f(x) w(x) dx = 0$$

by the definition of  $p_n$  and  $J(q \cdot p_n) = 0$  since  $p_n(x_i) = 0$ . We thus have

$$I(f) = I(r) = I\left(\sum_{i=1}^n r(x_i) l_i\right) = \sum_{i=1}^n r(x_i) I(l_i) = \sum_{i=1}^n r(x_i) \omega_i = J(r) = J(f).$$

b. For any weights  $\omega_i$ , let us take  $f(x) = (x - x_1)\dots(x - x_n)^2 \in \mathcal{P}_{2n}$ . Then  $J(f) = 0$  while  $I(f) \neq 0$  since  $f$  is not identical zero.

c. Since the function  $l_i^2(x) \in \mathcal{P}_{2n-2}$ , we have

$$\omega_i = J(l_i^2) = I(l_i^2) = \int_{-1}^1 l_i^2(x)w(x)dx > 0.$$

4. a. Yes. If  $g(x)$  is constant, then the solution is constant and attains the maximum inside  $\Omega_h$ . If  $g$  is not a constant function as the discrete function defined on  $\partial\Omega_h$ , then the answer is no. If  $U$  attains its maximum at some interior point, say  $(x_i, y_j) \in \Omega_h$ , then

$$\max U = U_{i,j} = \frac{1}{4}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}) \leq \max U.$$

It implies that

$$U_{i+1,j} = U_{i-1,j} = U_{i,j+1} = U_{i,j-1} = \max U$$

and thus  $U$  is constant on  $\partial\Omega_h$  which is a contradiction.

b. Applying the lemma to  $V = u - U$ , we get

$$\|u - U\|_{\infty, \Omega_h} \leq \frac{1}{8}\|\Delta_h(u - U)\| = \frac{1}{8}\|\Delta_h u\|_{\infty, \Omega_h}.$$

It suffices to prove

$$|(\Delta_h u)_{ij}| \leq Ch^2,$$

which can be derived from the fact

$$\frac{1}{h^2}[f(x-h) - 2f(x) + f(x+h)] - f''(x) = f^{(4)}(\xi)h^2.$$