There are 4 problems each worth 25 points. Partial credit will be given and 75 points are considered a full score for this exam. **Please show all your work to get full credit.**

**1.** Consider the finite difference approximation

$$\frac{U_j^{n+1} - 0.5(U_{j+1}^n + U_{j-1}^n)}{k} + \frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

to the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

where $U_j^n$ is the approximation to $u(x_j, t^n)$. Derive the consistency error of this scheme in an $O(h^l + k^m)$ form. In order to obtain it you can use the error estimates for the various difference operators involved in the scheme without proving them.

**2.** Consider the initial-boundary-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, t > 0 \tag{1}$$

$$u(0,t) = u(1,t) = 0, t > 0 \tag{2}$$

$$u(x,0) = f(x), \quad 0 < x < 1. \tag{3}$$

The following scheme is applied for discretization of this problem

$$\frac{U_j^{n+1} - U_j^n}{k} = 1/2\left(\frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{h^2} + \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}\right), \quad n = 1, 2, \ldots, N.$$

where $U_j^n = U(x_j, t_n)$ is the solution of the finite difference scheme.

(a) Using von Neumann analysis, study the stability of the scheme.

(b) What is the convergence error of this scheme? In order to obtain it you can use the error estimates for the various difference operators involved in the scheme without proving them.

**3.** Define the *region of absolute stability* of a single step method as

$$\mathcal{S} = \{\bar{h} \in \mathbb{C} | \lim_{n\to\infty} y_n = 0\},$$

where $y_n$ is the numerical solution to

$$y' = \lambda y, \quad y(0) = 1,$$

and $\bar{h} = h\lambda$ with step size $h$ and $\lambda \in \mathbb{C}$. For solving the equation $y' = f(t,y)$, consider the scheme

$$y_{n+1} = y_n + \frac{h}{2}(y_n' + y_{n+1}') + \frac{h^2}{12}(y_n'' - y_{n+1}'')$$

where $y_n' = f(t_n, y_n)$, and $y_n'' = \left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right)(t_n, y_n)$. Show that its region of absolute stability contains the entire negative real axis.

**4.** Let the coefficients $a_{ij}$ of the matrix $A$, of order $n$, satisfy

$$\nu_1 = \sum_{j=2}^{n} |a_{1j}|/|a_{11}| < 1,$$

$$\nu_i = \sum_{j<i} \frac{|a_{ij}|}{|a_{ii}|}\nu_j + \sum_{j>i} \frac{|a_{ij}|}{|a_{ii}|} < 1, \qquad i = 2, 3, ..., n.$$

Show that the Gauss-Seidel method converges for this matrix.