Lecture Notes for Systems of first order linear differential equations

Wen Shen

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NB! These notes are used by myself. They are provided to students as a supplement to the textbook. They can not substitute the textbook.

7.1: Introduction to systems of differential equations (0.5)

Given

\[ ay'' + by' + cy = g(t), \quad y(0) = \alpha, \quad y'(0) = \beta \]

we can do a variable change: let \( x_1 = y \) and \( x_2 = x_1' = y' \), then

\[
\begin{align*}
  x_1' &= x_2 \\
  x_2' &= y'' = \frac{1}{a}(g(t) - bx_2 - cx_1)
\end{align*}
\]

I.C.'s:

\[
\begin{align*}
  x_1(0) &= \alpha \\
  x_2(0) &= \beta
\end{align*}
\]

Observation: For any 2nd order equation, we can rewrite it into a system of 2 first order equations.

**Example 1.** Given

\[ y'' + 5y' - 10y = \sin t, \quad y(0) = 2, \quad y'(0) = 4 \]

Rewrite it into a system of first order equations: let \( x_1 = y \) and \( x_2 = y' = x_1' \), then

\[
\begin{align*}
  x_1' &= x_2 \\
  x_2' &= y'' = -5x_2 + 10x_1 + \sin t
\end{align*}
\]

I.C.’s:

\[
\begin{align*}
  x_1(0) &= 2 \\
  x_2(0) &= 4
\end{align*}
\]

We can do the same thing to any high order equations. For \( n \)-th order differential equation:

\[ y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}) \]
define the variable change:

\[ x_1 = y, \quad x_2 = y', \quad \cdots \quad x_n = y^{(n-1)} \]

we get

\[
\begin{aligned}
    x'_1 &= y' = x_2 \\
    x'_2 &= y'' = x_3 \\
    \vdots \\
    x'_{n-1} &= y^{(n-1)} = x_n \\
    x'_n &= y^{(n)} = F(t, x_1, x_2, \cdots, x_n)
\end{aligned}
\]

with corresponding source terms.
Reversely, we can convert a 1st order system into a high order equation.

**Example 2.** Given

\[
\begin{align*}
x_1' &= 3x_1 - 2x_2 \\
x_2' &= 2x_1 - 2x_2
\end{align*}
\]

Eliminate \(x_2\): the first equation gives

\[2x_2 = 3x_1 - x_1', \quad x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1'.\]

Plug this into second equation, we get

\[
\left(\frac{3}{2}x_1 - \frac{1}{2}x_1'\right)' = 2x_1 - 2x_2 = -x_1 + x_1'
\]

\[
\frac{3}{2}x_1' - \frac{1}{2}x_1'' = -x_1 + x_1'
\]

\[x_1'' - x_1' - 2x_1 = 0\]

with the initial conditions:

\[x_1(0) = 3, \quad x_1'(0) = 3x_1(0) - 2x_2(0) = 8.\]

This we know how to solve!

Definition of a solution: a set of functions \(x_1(t), x_2(t), \ldots, x_n(t)\) that satisfy the differential equations and the initial conditions.
7.2: Review of matrices (0.5)

A matrix of size \(m \times n\):
\[
A = \begin{pmatrix}
a_{1,1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots \\
a_{m,1} & \cdots & a_{m,n}
\end{pmatrix} = (a_{i,j}), \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.
\]

We consider only square matrices, i.e., \(m = n\), in particular for \(n = 2\) and \(3\).

Basic operations: \(A, B\) are two square matrices of size \(n \times n\).

- **Addition**: \(A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})\)
- **Scalar multiple**: \(\alpha A = (\alpha \cdot a_{ij})\)
- **Transpose**: \(A^T\) switch the \(a_{i,j}\) with \(a_{j,i}\). \((A^T)^T = A\).
- **Product**: For \(A \cdot B = C\), it means \(c_{i,j}\) is the inner product of \((i\text{th row of } A)\) and \((j\text{th column of } B)\). Example:
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
x & y \\
u & v
\end{pmatrix}
= \begin{pmatrix}
ax + bu & ay + bv \\
 cx + du & cy + dv
\end{pmatrix}
\]

We can express system of linear equations using matrix product.

**Example 1.**
\[
\begin{align*}
x_1 - x_2 + 3x_3 &= 4 \\
2x_1 + 5x_3 &= 0 \\
x_2 - x_3 &= 7
\end{align*}
\]

Can be expressed as:
\[
\begin{pmatrix}
1 & -1 & 3 \\
2 & 0 & 5 \\
0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
4 \\
0 \\
7
\end{pmatrix}
\]

**Example 2.**
\[
\begin{align*}
x_1' &= a(t)x_1 + b(t)x_2 + g_1(t) \\
x_2' &= c(t)x_1 + d(t)x_2 + g_2(t)
\end{align*}
\]

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}'
= \begin{pmatrix}
a(t) & b(t) \\
c(t) & d(t)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
+ \begin{pmatrix}
g_1(t) \\
g_2(t)
\end{pmatrix}
\]

Some properties:

- **Identity** \(I\): \(I = \text{diag}(1, 1, \cdots, 1)\), \(AI = IA = A\).
- **Determinant** \(\det(A)\):
\[
\det\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = ad - bc,
\det\begin{pmatrix}
a & b & c \\
u & v & w \\
x & y & z
\end{pmatrix} = avx + bwv + cuy - xvc - ywa - zub.
\]
- **Inverse** \(\text{inv}(A) = A^{-1}\): \(A^{-1}A = AA^{-1} = I\).
- The following statements are all equivalent: (1) \(A\) is invertible; (2) \(A\) is non-singular; (3) \(\det(A) \neq 0\); (4) row vectors in \(A\) are linearly independent; (5) column vectors in \(A\) are linearly independent. (6) All eigenvalues of \(A\) are non-zero.
7.3: Eigenvalues and eigenvectors (1)

Eigenvalues and eigenvectors of $A$ ($A$ is 2 × 2 or 3 × 3.)

$\lambda$: scalar value, $\vec{v}$: column vector, $\vec{v} \neq 0$.
If $A\vec{v} = \lambda \vec{v}$, then ($\lambda$, $\vec{v}$) is the (eigenvalue, eigenvector) of $A$. They are also called an eigen-pair of $A$.

Remark: If $\vec{v}$ is an eigenvector, then $\alpha \vec{v}$ for any $\alpha \neq 0$ is also an eigenvector, b/c $A(\alpha \vec{v}) = \alpha A \vec{v} = \alpha \lambda \vec{v} = \lambda (\alpha \vec{v})$.

How to find $(\lambda, \vec{v})$:

$$A\vec{v} - \lambda \vec{v} = 0,$$
$$\det(A - \lambda I) = 0.$$

We see that $\det(A - \lambda I)$ is a polynomial of degree 2 (or 3) of $\lambda$, and it is also called the characteristic polynomial of $A$. We need to find its roots.

Example 1: Find the eigenvalues and the eigenvectors of $A$ where 

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

Let’s first find the eigenvalues.

$$\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 1 \\ 4 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2 - 4 = 0, \quad \lambda_1 = -1, \lambda_2 = 3.$$

Now, let’s find the eigenvector $\vec{v}_1$ for $\lambda_1 = -1$: let $\vec{v}_1 = (a, b)^T$

$$(A - \lambda_1 I)\vec{v}_1 = 0, \quad \begin{pmatrix} 1 - (-1) & 1 \\ 4 & 1 - (-1) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so

$$2a + b = 0, \quad \text{choose } a = 1, \text{ then we have } b = -2, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Finally, we will compute the eigenvector $\vec{v}_2 = (c, d)^T$ for $\lambda_2 = 3$:

$$(A - \lambda_1 I)\vec{v}_2 = 0, \quad \begin{pmatrix} 1 - 3 & 1 \\ 4 & 1 - 3 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so

$$2c - d = 0, \quad \text{choose } c = 1, \text{ then we have } d = 2, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Example 2. Eigenvalues can be complex numbers.

$$A = \begin{pmatrix} 2 & -9 \\ 4 & 2 \end{pmatrix}.$$
Let’s first find the eigenvalues.

\[
\begin{vmatrix}
2 - \lambda & -9 \\
4 & 2 - \lambda
\end{vmatrix} = (2 - \lambda)^2 + 36 = 0, \quad \lambda_{1,2} = 2 \pm 6i
\]

We see that \(\lambda_2 = \bar{\lambda}_1\), complex conjugate. The same will happen to the eigenvectors, i.e., \(\vec{v}_1 = \bar{\vec{v}}_2\). So we need to only find one. Take \(\lambda_1 = 2 + 6i\), we compute \(\vec{v} = (v^1, v^2)^T:\)

\[
(A - \lambda_1 I)\vec{v} = 0, \quad \begin{pmatrix} -i6 & -9 \\ 4 & -i6 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = 0,
\]

\[-6i v^1 - 9 v^2 = 0, \quad \text{choose } v^1 = 1, \quad v^2 = \frac{2}{3}i,
\]

so

\[
\vec{v}_1 = \begin{pmatrix} 1 \\ -\frac{2}{3}i \end{pmatrix}, \quad \vec{v}_2 = \bar{\vec{v}}_1 = \begin{pmatrix} 1 \\ \frac{2}{3}i \end{pmatrix}.
\]

**Example** on repeated eigenvalues?
7.5: Homogeneous systems of two equations with constant coefficients.

We consider the following initial value problem:

\[
\begin{align*}
x_1' &= ax_1 + bx_2 \\
x_2' &= cx_1 + dx_2
\end{align*}
\]

I.C.'s: \[
\begin{align*}
x_1(0) &= \bar{x}_1 \\
x_2(0) &= \bar{x}_2
\end{align*}
\]

In matrix vector form:

\[
\vec{x}' = A\vec{x}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Claim: If \((\lambda, \vec{v})\) is an eigen-pair for \(A\), then \(\vec{z} = e^{\lambda t} \vec{v}\) is a solution to \(\vec{x}' = A\vec{x}\).

Proof.

\[
\vec{z}' = (e^{\lambda t} \vec{v})' = (e^{\lambda t})' \vec{v} = \lambda e^{\lambda t} \vec{v}
\]

\[
A\vec{z} = A(e^{\lambda t} \vec{v}) = e^{\lambda t} (A\vec{v}) = e^{\lambda t} \lambda \vec{v}
\]

Therefore \(\vec{z}' = A\vec{z}\) so \(\vec{z}\) is a solution.

Steps to solve the initial value problem:

- Step I: Find eigenvalues of \(A\): \(\lambda_1, \lambda_2\).
- Step II: Find the corresponding eigenvectors \(\vec{v}_1, \vec{v}_2\).
- Step III: Form two solutions: \(\vec{z}_1 = e^{\lambda_1 t} \vec{v}_1, \vec{z}_2 = e^{\lambda_2 t} \vec{v}_2\).
- Step IV: Check that \(\vec{z}_1, \vec{z}_2\) are linearly independent: the Wronskian

\[
W(\vec{z}_1, \vec{z}_2) = \det(\vec{z}_1, \vec{z}_2) \neq 0.
\]

(This step is usually OK in our problems.)

- Step V: Form the general solution: \(\vec{x} = c_1 \vec{z}_1 + c_2 \vec{z}_2\).

- If initial condition \(\vec{x}(0)\) is given, then use it to determine \(c_1, c_2\).

We will start with an example.

Example 1. Solve

\[
\vec{x}' = A\vec{x}, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.
\]

First, find out the eigenvalues of \(A\). By an example in 7.3, we have

\[
\lambda_1 = -1, \quad \lambda_2 = 3, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},
\]

\[7\]
So the general solution is
\[ \vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = c_1 e^{-t} \left( \begin{array}{c} 1 \\ -2 \end{array} \right) + c_2 e^{3t} \left( \begin{array}{c} 1 \\ 2 \end{array} \right). \]

Write it out in components:
\[
\begin{align*}
x_1(t) &= c_1 e^{-t} + c_2 e^{3t} \\
x_2(t) &= -2c_1 e^{-t} + 2c_2 e^{3t}.
\end{align*}
\]

Qualitative property of the solutions:

- **What happens when** \( t \to \infty \)?
  - If \( c_2 > 0 \), then \( x_1 \to \infty, x_2 \to \infty \).
  - If \( c_2 < 0 \), then \( x_1 \to -\infty, x_2 \to -\infty \).

Asymptotic relation between \( x_1, x_2 \): look at \( \frac{x_1}{x_2} \):
\[
\frac{x_1}{x_2} = \frac{c_1 e^{-t} + c_2 e^{3t}}{-2c_1 e^{-t} + 2c_2 e^{3t}}.
\]
As \( t \to \infty \), we have
\[
\frac{x_1}{x_2} = \frac{2c_2 e^{3t}}{2c_2 e^{3t}} = \frac{1}{2}
\]
This means, \( x_1 \to 2x_2 \) asymptotically.

- **What happens when** \( t \to -\infty \)?
  - Looking at \( \frac{x_1}{x_2} \), we see as \( t \to -\infty \) we have
\[
\frac{x_1}{x_2} = \frac{c_1 e^{-t}}{-2c_1 e^{-t}} = -\frac{1}{2},
\]
which means, \( x_1 \to -2x_2 \) asymptotically as \( t \to -\infty \).

**Phase portrait.** is the trajectories of various solutions in the \( x_2 - x_1 \) plane.

- Since \( A \) is non-singular, then \( \vec{x} = \vec{0} \) is the only critical point such that \( \vec{x}' = A \vec{x} = 0 \).

- If \( c_1 = 0 \), then \( \frac{x_1}{x_2} = \frac{c_2 e^{3t}}{2c_2 e^{3t}} = \frac{1}{2} \), so the trajectory is a straight line \( x_1 = 2x_2 \).
  
  Note that this is exactly the direction of \( \vec{v}_2 \).
  
  Since \( \lambda_2 = 3 > 0 \), the trajectory is going away from 0.

- If \( c_2 = 0 \), then \( \frac{x_1}{x_2} = \frac{c_1 e^{-t}}{-2c_1 e^{-t}} = -\frac{1}{2} \), so the trajectory is another straight line \( x_1 = -2x_2 \).
  
  Note that this is exactly the direction of \( \vec{v}_1 \).

  Since \( \lambda_2 = -1 < 0 \), the trajectory is going towards 0.
• For general cases where $c_1, c_2$ are not 0, the trajectories should start (asymptotically) from line $x_1 = -2x_2$, and goes to line $x_1 = 2x_2$ asymptotically as $t$ grows.

**Definition:** If $A$ has two real eigenvalues of opposite signs, the origin (critical point) is called a **saddle point**. A saddle point is unstable.

**Tips for drawing phase portrait for saddle point:** only need the eigenvalues and eigenvectors!

**General case:** If two eigenvalues of $A$ are $\lambda_1 < 0$ and $\lambda_2 > 0$, with two corresponding eigenvectors $\vec{v}_1, \vec{v}_2$. To draw the phase portrait, we follow these guidelines:

• The general solution is 
  $$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2.$$ 

• If $c_1 = 0$, then the solution is $\vec{x} = c_2 e^{\lambda_2 t} \vec{v}_2$. We see that the solution vector is a scalar multiple of $\vec{v}_2$. This means a line parallel to $\vec{v}_2$ through the origin is a trajectory. Since $\lambda_2 > 0$, solutions $|\vec{x}| \to \infty$ along this line, so the arrows are pointing away from the origin.

• The similar other half: if $c_2 = 0$, then the solution is $\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1$. We see that the solution vector is a scalar multiple of $\vec{v}_1$. This means a line parallel to $\vec{v}_1$ through the origin is a trajectory. Since $\lambda_1 < 0$, solutions approach 0 along this line, so the arrows are pointing toward the origin.

• Now these two line cut the plane into 4 regions. We need to draw at least one trajectory in each region. In the region, we have the general case, i.e., $c_1 \neq 0$ and $c_2 \neq 0$. We need to know the asymptotic behavior. We have
  $$t \to \infty, \quad \Rightarrow \quad \vec{x} \to c_2 e^{\lambda_2 t} \vec{v}_2$$
  $$t \to -\infty, \quad \Rightarrow \quad \vec{x} \to c_1 e^{\lambda_1 t} \vec{v}_1$$
We see these are exactly the two straight lines we just made. This means, all trajectories come from the direction of $\vec{v}_1$, and will approach $\vec{v}_2$ as $t$ grows. See the plot below.

Example 2. Suppose we know the eigenvalues and eigenvectors of $A$:

$$
\lambda_1 = 3, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_1 = -3, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

Then the phase portrait looks like this:
If the two real distinct eigenvalues have the same sign, the situation is quite different.

**Example 3.** Consider the homogeneous system

\[ \vec{x}' = A\vec{x}, \quad A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}. \]

Find the general solution and sketch the phase portrait.

- **Eigenvalues of** \( A \):
  \[ \det(A - \lambda I) = \det \begin{pmatrix} -3 - \lambda & 2 \\ 1 & -2 - \lambda \end{pmatrix} = (-3 - \lambda)(-2 - \lambda) - 2 = \lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4) = 0, \]
  So \( \lambda_1 = -1, \lambda_2 = -4 \). (Two eigenvalues are both negative!)

- **Find the eigenvalue for** \( \lambda_1 \). Call it \( \vec{v}_1 = (a, b)^T \),
  \[ (A - \lambda_1 I)\vec{v}_1 = \begin{pmatrix} -3 + 1 & 2 \\ 1 & -2 + 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]
  This gives \( a = b \). Choose it to be 1, we get \( \vec{v}_1 = (1, 1)^T \).

- **Find the eigenvalue for** \( \lambda_2 \). Call it \( \vec{v}_2 = (c, d)^T \),
  \[ (A - \lambda_2 I)\vec{v}_2 = \begin{pmatrix} -3 + 4 & 2 \\ 1 & -2 + 4 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]
  This gives \( c + 2d = 0 \). Choose \( d = 1 \), then \( c = -2 \). So \( \vec{v}_2 = (-2, 1)^T \).

- **General solution is**
  \[ \vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \]

  Write it out in components:
  \[
  \begin{cases}
  x_1(t) = c_1 e^{-t} - 2c_2 e^{-4t} \\
  x_2(t) = c_1 e^{-t} + c_2 e^{-4t}
  \end{cases}
  \]

**Phase portrait:**

- If \( c_1 = 0 \), then \( \vec{x} = c_2 e^{\lambda_2 t}\vec{v}_2 \), so the straight line through the origin in the direction of \( \vec{v}_2 \) is a trajectory. Since \( \lambda_2 < 0 \), the arrows point toward the origin.

- If \( c_2 = 0 \), then \( \vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 \), so the straight line through the origin in the direction of \( \vec{v}_1 \) is a trajectory. Since \( \lambda_1 < 0 \), the arrows point toward the origin.
For the general case, when \( c_1 \neq 0 \) and \( c_2 \neq 0 \), we have

\[
\begin{align*}
  t \to -\infty, & \implies \vec{x} \to 0, & & \vec{x} \to c_2 e^{\lambda_2 t} \vec{v}_2 \\
  t \to \infty, & \implies |\vec{x}| \to \infty, & & \vec{x} \to c_1 e^{\lambda_1 t} \vec{v}_1
\end{align*}
\]

So all trajectories come into the picture in the direction of \( \vec{v}_2 \), and approach the origin in the direction of \( \vec{v}_1 \). See the plot below.

In the previous example, if \( \lambda_1 > 0, \lambda_2 > 0 \), say \( \lambda_1 = 1 \) and \( \lambda_2 = 4 \), and \( \vec{v}_1, \vec{v}_2 \) are the same, then the phase portrait will look the same, but with all arrows going away from 0.

**Definition:** If \( \lambda_1 \neq \lambda_2 \) are real with the same sign, the critical point \( \vec{x} = 0 \) is called a **node**.

If \( \lambda_1 > 0, \lambda_2 > 0 \), this node is called a **source**.

If \( \lambda_1 < 0, \lambda_2 < 0 \), this node is called a **sink**.

A sink is stable, and a source is unstable.

**Example 4.** (Source node) Suppose we know the eigenvalues and eigenvectors of \( A \) are

\[
\lambda_1 = 3, \lambda_2 = 4, \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.
\]

(1) Find the general solution for \( \vec{x}' = A\vec{x} \), (2) Sketch the phase portrait.

Solution: (1) The general solution is simple, just use the formula

\[
\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.
\]
(2) Phase portrait: Since $\lambda_2 > \lambda_1$, then the solution approach $v_2$ as time grows. As $t \to -\infty$, $\vec{x} \to c_1 e^{\lambda_1 t} \vec{v}_1$. See the plot below.
7.6: Complex eigenvalues

If $A$ has two complex eigenvalues, they will be a pair of complex conjugate numbers, say $\lambda_{1,2} = \alpha \pm i\beta$, $\beta \neq 0$.

The two corresponding eigenvectors will also be complex conjugate, i.e, $\vec{v}_1 = \bar{\vec{v}}_2$.

We have two solutions

$$
\vec{z}_1 = e^{\lambda t} \vec{v}_1, \quad \vec{z}_2 = e^{\lambda t} \vec{v}_2.
$$

They are complex-valued functions, and they also are complex conjugate. We seek real-valued solutions. By the principle of superposition,

$$
\vec{y}_1 = \frac{1}{2}(\vec{z}_1 + \vec{z}_2) = \text{Re}(\vec{z}_1), \quad \vec{y}_2 = \frac{1}{2i}(\vec{z}_1 - \vec{z}_2) = \text{Im}(\vec{z}_1)
$$

are also two solutions, and they are real-valued.

One can show that they are linearly independent, so they form a set of fundamental solutions. The general solution is then $\vec{x} = c_1 \vec{y}_1 + c_2 \vec{y}_2$.

Now let's derive the formula for the general solution. We have two eigenvalues: $\lambda$ and $\bar{\lambda}$, two eigenvectors: $\vec{v}$ and $\bar{\vec{v}}$, which we can write

$$
\lambda = \alpha + i\beta, \quad \vec{v} = \vec{v}_r + i\vec{v}_i.
$$

One solution can be written

$$
\vec{z} = e^{\lambda t} \vec{v} = e^{(\alpha + i\beta)t}(\vec{v}_r + i\vec{v}_i)e^{\alpha t}(\cos \beta t + i \sin \beta t)(\vec{v}_r + i\vec{v}_i)
$$

$$
= e^{\alpha t}(\cos \beta t \cdot \vec{v}_r - \sin \beta t \cdot \vec{v}_i + i(\sin \beta t \cdot \vec{v}_r + \cos \beta t \cdot \vec{v}_i)).
$$

The general solution is

$$
\vec{x} = c_1 e^{\alpha t}(\cos \beta t \cdot \vec{v}_r - \sin \beta t \cdot \vec{v}_i) + c_2 e^{\alpha t}(\sin \beta t \cdot \vec{v}_r + \cos \beta t \cdot \vec{v}_i).
$$

Notice now if $\alpha = 0$, i.e., we have pure imaginary eigenvalues. The $\vec{x}$ is a harmonic oscillation, which is a periodic function. This means in the phase portrait all trajectories are closed curves.

**Example 1.** (pure imaginary eigenvalues.) Find the general solution and sketch the phase portrait of the system:

$$
\vec{x}' = A\vec{x}, \quad A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}.
$$

First find the eigenvalues of $A$:

$$
\det(A - \lambda I) = \lambda^2 + 4 = 0, \quad \lambda_{1,2} = \pm 2i.
$$

Eigenvectors: need to find one $\vec{v} = (a, b)^T$ for $\lambda = 2i$:

$$
(A - \lambda I)\vec{v} = 0,
$$

$$
\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$
\[ a - 2ib = 0, \quad \text{choose } b = 1, \text{ then } a = 2i, \]

then

\[ \vec{v} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \]

The general solution is

\[ \vec{x} = c_1 (\cos 2t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin 2t \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix}) + c_2 (\sin 2t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cos 2t \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix}). \]

Write out the components, we get

\[ x_1(t) = -2c_1 \sin 2t + 2c_2 \cos 2t \]
\[ x_2(t) = c_1 \cos 2t + c_2 \sin 2t. \]

**Phase portrait:**

- \( \vec{x} \) is a periodic function, so all trajectories are closed curves around the origin.

- They do not intersect with each other. This follows from the uniqueness of the solution.

- They are ellipses. Because we have the relation: \( (x_1/2)^2 + (x_2)^2 = \text{constant}. \)

- The arrows are pointing either clockwise or counter clockwise, determined by \( A \). In this example, take \( \vec{x} = (1, 0)^T \), a point on the \( x_1 \)-axis. By the differential equations, we get \( \vec{x}' = A\vec{x} = (0, 1)^T \), which is a vector pointing upward. So the arrows are counter-clockwise.

See plot below.

**Definition.** The origin in this case is called a *center*. A center is stable (b/c solutions don’t blow up), but is not asymptotically stable (b/c solutions don’t approach the origin as time goes).
If the complex eigenvalues have non-zero real part, the situation is still different.

**Example 2.** Consider the system

\[
\vec{x}' = A \vec{x}, \quad A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}.
\]

First, we compute the eigenvalues:

\[
det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) + 8 = \lambda^2 - 2\lambda + 5 = 0, \quad \lambda_{1,2} = 1 \pm 2i, \quad \alpha = 1, \beta = 2.
\]

Eigenvectors: need to compute only one \( \vec{v} = (a, b)^T \). Take \( \lambda = 1 + 2i \),

\[
(A - \lambda I)\vec{v} = \begin{pmatrix} 2 - 2i \\ 4 \\ -2 - 2i \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (2 - 2i)a - 2b = 0.
\]

Choosing \( a = 1 \), then \( b = 1 - i \), so

\[
\vec{v} = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]

So the general solution is:

\[
\vec{x} = c_1 e^t \left[ \cos 2t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + c_2 e^t \left[ \sin 2t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \cos 2t \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\
= c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}.
\]

**Phase portrait.** Solution is growing oscillation due to the \( e^t \). If this term is not present, (i.e., the eigenvalues would be pure imaginary), then the solutions are perfect oscillations, whose trajectory would be closed curves around origin, as the center. But with the \( e^t \) term, we will get spiral curves. Since \( \alpha = 1 > 0 \), all arrows are pointing away from the origin.

To determine the direction of rotation, we need to go back to the original equation and take a look at the directional field.

Consider the point \((x_1 = 1, x_2 = 0)\), then \( \vec{x}' = A \vec{x} = (3, 4)^T \). The arrow should point up with slope \( 4/3 \).

At the point \( \vec{x} = (0, 1)^T \), we have \( \vec{x}' = (-2, -1)^T \).

Therefore, the spirals are rotating counter clockwise. We don’t stress on the exact shape of the spirals. See plot below.
In this case, the origin (the critical point) is called the **spiral point**. The origin in this example is an unstable critical point since $\alpha > 0$.

**Remark:** If $\alpha < 0$, then all arrows will go towards the origin. The origin will be a stable critical point. We skip the example for this case.
7.8: Repeated eigenvalues

Here we study the case where the two eigenvalues are the same, say \( \lambda_1 = \lambda_2 = \lambda \). This can happen, say if we have

\[
A = \begin{pmatrix}
1 & -1 \\
1 & 3
\end{pmatrix}.
\]

Then

\[
det(A - \lambda I) = det\left(\begin{pmatrix}
1 - \lambda & -1 \\
1 & 3 - \lambda
\end{pmatrix}\right) = (1 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 4\lambda + 3 + 1 = (\lambda - 2)^2 = 0,
\]

so \( \lambda_1 = \lambda_2 = 2 \). And we can find only one eigenvector \( \vec{v} = (a, b)^T \)

\[
(A - \lambda I)\vec{v} = \begin{pmatrix}
-1 & -1 \\
1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
a \\
b
\end{pmatrix} = 0, \quad a + b = 0.
\]

Choosing \( a = 1 \), then \( b = -1 \), and we find \( \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \). Then, one solution is:

\[
\vec{z}_1 = e^{\lambda t} \vec{v} = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

We need to find a second solution. Let’s try \( \vec{z}_2 = te^{\lambda t} \vec{v} \). We have

\[
\vec{z}' = e^{\lambda t} \vec{v} + \lambda te^{\lambda t} \vec{v} = (1 + \lambda t)e^{\lambda t} \vec{v}
\]

\[
A\vec{z}_2 = Ate^{\lambda t} \vec{v} = te^{\lambda t}(A\vec{v}) = te^{\lambda t} \lambda \vec{v} = \lambda te^{\lambda t} \vec{v}
\]

If \( \vec{z}_2 \) is a solution, we must have

\[
\vec{z}' = A\vec{z} \quad \Rightarrow \quad 1 + \lambda t = \lambda t
\]

which doesn’t work.

Try something else: \( \vec{z}_2 = te^{\lambda t} \vec{v} + \vec{\eta}e^{\lambda t} \). (here \( \vec{\eta} \) is a constant vector to be determined later).

Then

\[
\vec{z}_2 = (1 + \lambda t)e^{\lambda t} \vec{v} + \lambda \vec{\eta}e^{\lambda t} = \lambda te^{\lambda t} \vec{v} + e^{\lambda t}(\vec{v} + \lambda \vec{\eta})
\]

\[
A\vec{z}_2 = \lambda te^{\lambda t} \vec{v} + A\vec{\eta}e^{\lambda t}.
\]

Since \( \vec{z}_2 \) is a solution, we must have \( \vec{z}' = A\vec{z} \). Comparing terms, we see we must have

\[
\vec{v} + \lambda \vec{\eta} = A\vec{\eta}, \quad (A - \lambda I)\vec{\eta} = \vec{v}.
\]

This is what one uses to solve for \( \vec{\eta} \). Such an \( \vec{\eta} \) is called a generalized eigenvector corresponding to the eigenvalue \( \lambda \).

Back to the original problem, to compute this \( \vec{\eta} \), we plug in \( A \) and \( \lambda \), and get

\[
\begin{pmatrix}
-1 & -1 \\
1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix} = \begin{pmatrix} 1 \\
-1
\end{pmatrix}, \quad \eta_1 + \eta_2 = -1.
\]
We can choose $\eta_1 = 0$, then $\eta_2 = -1$, and so $\vec{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

So the general solution is

$$\vec{x} = c_1 \vec{z}_1 + c_2 \vec{z}_2 = c_1 e^{t \vec{v}} + c_2 (te^{t \vec{v}} + e^{t \vec{v}} \vec{\eta}) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \left[ te^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right].$$

Phase portrait:

- As $t \to \infty$, we have $|\vec{x}| \to \infty$ unbounded.
- As $t \to -\infty$, we have $\vec{x} \to 0$.
- If $c_2 = 0$, then $\vec{x} = c_1 e^{t \vec{v}}$, so the line through the origin in the direction of $\vec{v}$ is a trajectory. Since $\lambda > 0$, the arrows point away from the origin.
- If $c_1 = 0$, then $\vec{x} = c_2 (te^{t \vec{v}} + e^{t \vec{v}} \vec{\eta})$. For this solution, as $t \to \infty$, the dominant term in $\vec{x}$ is $te^{t \vec{v}}$. This means the solution approach the direction of $\vec{v}$. On the other hand, as $t \to -\infty$, the dominant term in $\vec{x}$ is still $te^{t \vec{v}}$. This means the solution approach the direction of $\vec{v}$. But, due to the change of sign of $t$, the $\vec{x}$ will change direction and point toward the opposite direction as when $t \to \infty$.

How does it turn? We need to go back to the system and check the directional field. At $\vec{x} = (1, 0)$, we have $\vec{x}' = (1, 1)^T$, and at $\vec{x} = (0, 1)$, we have $\vec{x}' = (-1, 3)^T$. There it turns kind of counter clockwise. See figure below.

- For the general case, with $c_1 \neq 0$ and $c_2 \neq 0$, a similar thing happens. As $t \to \infty$, the dominant term in $\vec{x}$ is $te^{t \vec{v}}$. This means the solution approach the direction of $\vec{v}$. As $t \to -\infty$, the dominant term in $\vec{x}$ is still $te^{t \vec{v}}$. This means the solution approach the direction of $\vec{v}$. But, due to the change of sign of $t$, the $\vec{x}$ will change direction and point toward the opposite direction as when $t \to \infty$. See plot below.

Remark: If $\lambda < 0$, the phase portrait looks the same except with reversed arrows.
**Definition.** If $A$ has repeated eigenvalues, the origin is called a *improper node*. It is stable if $\lambda < 0$, and unstable if $\lambda > 0$.

**Example 1.** Find the general solution to the system $\vec{x}' = \begin{pmatrix} 2 & -2 \\ 0.5 & 4 \end{pmatrix} \vec{x}$.

We start with finding the eigenvalues:

$$\det(A - \lambda I) = (2 - \lambda)(4 - \lambda) + 1 = \lambda^2 - 6\lambda + 8 + 1 = (\lambda - 3)^2 = 0, \quad \lambda_1 = \lambda_2 = \lambda = 3$$

We see we have double eigenvalue. The corresponding eigenvector $\vec{v} = (a, b)^T$

$$(A - \lambda I)\vec{v} = \begin{pmatrix} 2 - 3 & -2 \\ 0.5 & 4 - 3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0.5 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

So we must have $a + 2b = 0$. Choose $a = 2$, then $b = -1$, and we get $\vec{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. To find the generalized eigenvector $\vec{\eta}$, we solve

$$(A - \lambda I)\vec{\eta} = \vec{v}, \quad \begin{pmatrix} -1 \\ 0.5 \end{pmatrix} \cdot \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$ 

This gives us one relation $-\eta_1 - 2\eta_2 = 2$. Choose $\eta_1 = 0$, then we have $\eta_2 = -1$, and so $\vec{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The general solution is

$$\vec{x} = c_1 e^{\lambda t} \vec{v} + c_2 (te^{\lambda t} \vec{v} + e^{\lambda t} \vec{\eta}) = c_1 e^{3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \left[ te^{3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + e^{3t} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right].$$

Just for fun, I include the phase portrait below.

![Phase portrait](image.png)

The origin is an improper node which is unstable.