Lecture Notes Math 250:  
Ordinary Differential Equations

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NB! These notes are used by myself. They are provided to students as a supplement to the textbook. They can not substitute the textbook.

Chapter 2: First order Differential Equations

We consider the equation

\[ \frac{dy}{dt} = f(t, y) \]

Overview:

- Two special types of equations: linear, and separable;
- Linear vs. nonlinear;
- modeling;
- autonomous equations.
2.1: Linear equations; Method of integrating factors

The function $f(t, y)$ is a linear function in $y$, i.e., we can write

$$f(t, y) = -p(t)y + g(t).$$

So we will study the equation

$$y' + p(t)y = g(t).$$

We introduce the method of integrating factors (due to Leibniz): We multiply equation (A) by a function $\mu(t)$ on both sides

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

The function $\mu$ is chosen such that the equation is integrable, meaning the LHS (Left Hand Side) is the derivative of something. In particular, we require:

$$\mu(t)y' + \mu(t)p(t)y = (\mu(t)y)', \quad \Rightarrow \quad \mu(t)y' + \mu(t)p(t)y = \mu(t)y' + \mu'(t)y$$

which requires

$$\mu'(t) = \frac{d\mu}{dt} = \mu(t)p(t), \quad \Rightarrow \quad \frac{d\mu}{\mu} = p(t)\, dt$$

Integrating both sides

$$\ln \mu(t) = \int p(t)\, dt$$

which gives a formula to compute $\mu$

$$\mu(t) = \exp\left(\int p(t)\, dy\right).$$

Therefore, this $\mu$ is called the integrating factor. Putting back into equation (A), we get

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t), \quad \mu(t)y = \int \mu(t)g(t)\, dt + c$$
which give the formula for the solution

\[ y(t) = \frac{1}{\mu(t)} \left[ \int \mu(t)g(t) \, dt + c \right], \quad \text{where} \quad \mu(t) = \exp \left( \int p(t) \, dt \right). \]

**Example 1.** Solve \( y' + ay = b \) \((a \neq 0)\).

**Answer.** We have \( p(t) = a \) and \( g(t) = b \). So

\[ \mu = \exp(\int a \, dt) = e^{at} \]

so

\[ y = e^{-at} \int e^{at}b \, dt = e^{-at} \left( \frac{b}{a} e^{at} + c \right) = \frac{b}{a} + ce^{-at}, \]

where \( c \) is an arbitrary constant.

**Example 2.** Solve \( y' + y = e^{2t} \).

**Answer.** We have \( p(t) = 1 \) and \( g(t) = e^{2t} \). So

\[ \mu(t) = \exp(\int 1 \, dt) = e^{t} \]

and

\[ y(t) = e^{-t} \int e^{t} e^{2t} \, dt = e^{t} \int e^{3t} \, dt = e^{t} \left( \frac{1}{3} e^{3t} + c \right) = \frac{1}{3} e^{2t} + ce^{-t}. \]

**Example 3.** Solve

\[ (1 + t^2)y' + 4ty = (1 + t^2)^{-2}, \quad y(0) = 1. \]

**Answer.** First, let’s rewrite the equation into the normal form

\[ y' + \frac{4t}{1 + t^2}y = (1 + t^2)^{-3}, \]

so

\[ p(t) = \frac{4t}{1 + t^2}, \quad g(t) = (1 + t^2)^{-3}. \]
Then
\[ \mu(t) = \exp\left( \int p(t) \, dt \right) = \exp\left( \int \frac{4t}{1 + t^2} \, dt \right) = \exp(2 \ln(1 + t^2)) = \exp(\ln(1 + t^2)^2) = (1 + t^2)^2. \]

Then
\[ y = (1+t^2)^{-2} \int (1+t^2)^2(1+t^2)^{-3} \, dt = (1+t^2)^{-2} \int (1+t^2)^{-1} \, dt = \frac{\arctan t + c}{(1+t^2)^2}. \]

By the IC \( y(0) = 1 \):
\[ y(0) = \frac{0 + c}{1} = c = 1, \quad \Rightarrow \quad y(t) = \frac{\arctan t + 1}{(1+t^2)^2}. \]

Example 4. Solve \( ty' - y = t^2e^{-t}, \quad (t > 0) \).

Answer. Rewrite it into normal form
\[ y' - \frac{1}{t}y = te^{-t} \]
so
\[ p(t) = -1/t, \quad g(t) = te^{-t}. \]
We have
\[ \mu(t) = \exp(\int (-1/t) \, dt) = \exp(-\ln t) = \frac{1}{t} \]
and
\[ y(t) = t \int \frac{1}{t} e^{-t} \, dt = t \int e^{-t} \, dt = t(-e^{-t} + c) = -te^{-t} + ct. \]

Example 5. Solve \( y - \frac{1}{2}y = e^{-t} \), with \( y(0) = a \), and discussion how the behavior of \( y \) as \( t \to \infty \) depends on the initial value \( a \).

Answer. Let’s solve it first. We have
\[ \mu = e^{-\frac{1}{2}t} \]
so
\[ y = e^{\frac{1}{2}t} \int e^{-\frac{1}{2}t} e^{-t} \, dt = e^{\frac{1}{2}t} \int e^{-\frac{3}{2}t} \, dt = e^{\frac{1}{2}t}(-\frac{3}{4}e^{-\frac{3}{2}t} + c). \]
Plug in the IC to find $c$

$$y(0) = e^0(-\frac{3}{4} + c) = a, \quad c = a + \frac{3}{4}$$

so

$$y(t) = e^{\frac{t}{2}} \left(-\frac{3}{4}e^{-\frac{t}{2}} + a + \frac{3}{4}\right) = -\frac{3}{4}e^{-t} + (a + \frac{3}{4})e^{t/3}.$$  

To see the behavior of the solution, we see that it contains two terms. The first term $e^{-t}$ goes to 0 as $t$ grows. The second term $e^{t/3}$ goes to $\infty$ as $t$ grows, but the constant $a + \frac{3}{4}$ is multiplied on it. So we have

- If $a + \frac{3}{4} = 0$, i.e., if $a = -\frac{3}{4}$, we have $y \to 0$ as $t \to \infty$;
- If $a + \frac{3}{4} > 0$, i.e., if $a > -\frac{3}{4}$, we have $y \to \infty$ as $t \to \infty$;
- If $a + \frac{3}{4} < 0$, i.e., if $a < -\frac{3}{4}$, we have $y \to -\infty$ as $t \to \infty$;

**Example 6.** Solve $ty' + 2y = 4t^2$, $y(1) = 2$.

**Answer.** Rewrite the equation first

$$y' + \frac{2}{t}y = 4t, \quad (t \neq 0)$$

So $p(t) = 2/t$ and $g(t) = 4t$. We have

$$\mu(t) = \exp\left(\int 2/t \, dt\right) = \exp(2 \ln t) = t^2$$

and

$$y(t) = t^{-2} \int 4t \cdot t^2 \, dy = t^{-2}(t^4 + c)$$

By IC $y(1) = 2$,

$$y(1) = 1 + c = 2, \quad c = 1$$

we get the solution:

$$y(t) = t^2 + \frac{1}{t^2}, \quad t > 0.$$  

Note the condition $t > 0$ comes from the fact that the initial condition is given at $t = 1$, and we require $t \neq 0$. 

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In the graph below we plot several solutions in the $t - y$ plane, depending on initial data. The one for our solution is plotted with dashed line where the initial point is marked with a ‘$x$’.
2.2: Separable Equations

We study first order equations that can be written as
\[ \frac{dy}{dx} = f(x, y) = \frac{M(x)}{N(y)} \]
where \( M(x) \) and \( N(y) \) are suitable functions of \( x \) and \( y \) only. Then we have
\[ N(y) \, dy = M(x) \, dx, \quad \Rightarrow \quad \int N(y) \, dy = \int M(x) \, dx \]
and we get implicitly defined solutions of \( y(x) \).

Example 1. Consider
\[ \frac{dy}{dx} = \frac{\sin x}{1 - y^2}. \]
We can separate the variables:
\[ \int (1 - y^2) \, dy = \int \sin x \, dx, \quad \Rightarrow \quad y - \frac{1}{3} y^3 = -\cos x + c. \]
If one has IC as \( y(\pi) = 2 \), then
\[ 2 - \frac{1}{3} \cdot 2^3 = -\cos \pi + c, \quad \Rightarrow \quad c = \frac{5}{3}, \]
so the solution \( y(x) \) is implicitly given as
\[ y - \frac{1}{3} y^3 + \cos x + \frac{5}{3} = 0. \]

Example 2. Find the solution in explicit form for the equation
\[ \frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y + 1)}, \quad y(0) = -1. \]
Answer. Separate the variables
\[ \int 2(y - 1) \, dy = \int (3x^2 + 4x + 2) \, dx, \quad \Rightarrow \quad (y - 1)^2 = x^3 + 2x^2 + 2x + c \]
Set in the IC $y(0) = -1$, i.e., $y = -1$ when $x = 0$, we get

$(-1 - 1)^2 = 0 + c, \quad c = 4, \quad (y - 1)^2 = x^3 + 2x^2 + 2x + 4$.

In explicitly form, one has two choices:

$$y(t) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$ 

To determine which sign is the correct one, we check again by the initial condition:

$$y(0) = 1 \pm \sqrt{4} = 1 \pm 2 = -1$$

We see we must choose the ‘-’ sign. The solution in explicitly form is:

$$y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$ 

On which interval will this solution be defined?

$$x^3 + 2x^2 + 2x + 4 \geq 0, \quad \Rightarrow \quad x^2(x + 2) + 2(x + 2) \geq 0$$

$$\Rightarrow \quad (x^2 + 2)(x + 2) \geq 0, \quad \Rightarrow \quad x \geq -2.$$ 

We can also argue that when $x = -2$, we have $y = 1$. At this point $|dy/dx| \to \infty$, therefore solution can not be defined at this point.

The plot of the solution is given below, where the initial data is marked with ‘x’. We also include the solution with the ‘+’ sign, using dotted line.
Example 3. Solve $y' = 3x^2 + 3x^2y^2$, $y(0) = 0$, and find the interval where the solution is defined.

**Answer.** Let’s first separate the variables.

$$\frac{dy}{dx} = 3x^2(1 + y^2), \quad \Rightarrow \int \frac{1}{1 + y^2}dy = \int 3x^2dx, \quad \Rightarrow \arctan y = x^3 + c.$$ 

Set in the IC:

$$\arctan 0 = 0 + c, \quad \Rightarrow \quad c = 0$$

we get the solution

$$\arctan y = x^3, \quad \Rightarrow \quad y = \tan(x^3).$$

Since the initial data is given at $x = 0$, i.e., $x^3 = 0$, and tan is defined on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$-\frac{\pi}{2} < x^3 < \frac{\pi}{2}, \quad \Rightarrow \quad -\left[\frac{\pi}{2}\right]^{1/3} < x < \left[\frac{\pi}{2}\right]^{1/3}.$$ 

Example 4. Solve

$$y' = \frac{1 + 3x^2}{3y^2 - 6y}, \quad y(0) = 1$$

and identify the interval where solution is valid.

**Answer.** Separate the variables

$$\int (3y^2 - 6y)dy = \int (1 + 3x^2)dx \quad y^3 - 3y^2 = x + x^3 + c.$$ 

Set in the IC: $x = 0$, $y = 1$, we get

$$1 - 3 = c, \quad \Rightarrow \quad c = -2,$$

Then,

$$y^3 - 3y^2 = x^3 - x - 2.$$ 

Note that solution is given in implicitly form.

To find the valid interval of this solution, we note that $y'$ is not defined is $3y^2 - 6y = 0$, i.e., when $y = 0$ or $y = 2$. These are the two so-called “bad
points” where you can not define the solution. To find the corresponding values of \( x \), we use the solution expression:

\[
y = 0 : \quad x^3 + x - 2 = 0, \quad \Rightarrow \quad (x^2 + x + 2)(x - 1) = 0, \quad \Rightarrow \quad x = 1
\]

and

\[
y = 2 : \quad x^3 + x - 2 = -4, \quad \Rightarrow \quad x^3 + x + 2 = 0, \quad \Rightarrow \quad (x^2 - x + 2)(x + 1) = 0, \quad \Rightarrow \quad x = -1
\]

(Note that we used the facts \( x^2 + x + 2 \neq 0 \) and \( x^2 - x + 2 \neq 0 \) for all \( x \).)

Draw the real line and work on it as following:

\[
\begin{array}{ccccccc}
\text{x} & \text{-2} & \text{-1} & \text{0} & \text{1} & \text{2} \\
\end{array}
\]

Therefore the interval is \(-1 < x < 1\).
2.4: Differences between linear and nonlinear equations

We will take this chapter before the modeling (ch. 2.3).

For a linear equation

\[ y' + p(t)y = g(t), \quad y(t_0) = y_0, \]

we have the following existence and uniqueness theorem.

**Theorem.** If \( p(t) \) and \( g(t) \) are continuous and bounded on an open interval containing \( t_0 \), then it has an unique solution on that interval.

**Example 1.** Find the largest interval where the solution can be defined for the following problems.

(A). \( ty' + y = t^3, \quad y(-1) = 3. \)

**Answer.** Rewrite: \( y' + \frac{1}{t}y = t^2 \), so \( t \neq 0 \). Since \( t_0 = -1 \), the interval is \( t < 0 \).

(B). \( ty' + y = t^3, \quad y(1) = -3. \)

**Answer.** The equation is same as (A), so \( t \neq 0 \). \( t_0 = 1 \), the interval is \( t > 0 \).

(C). \( (t - 3)y' + (\ln t)y = 2t, \quad y(1) = 2 \)

**Answer.** Rewrite: \( y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3} \), so \( t \neq 3 \) and \( t > 0 \) for the \( \ln \) function. Since \( t_0 = 1 \), the interval is then \( 0 < t < 3 \).

(D). \( y' + (\tan t)y = \sin t, \quad y(\pi) = 100. \)

**Answer.** Since \( t_0 = \pi \), and for \( \tan t \) to be defined we must have \( t \neq \frac{2k+1}{2}, \quad k = \pm 1, \pm 2, \ldots \). So the interval is \( \frac{\pi}{2} < t < \frac{3\pi}{2} \).

For non-linear equation

\[ y' = f(t, y), \quad y(t_0) = y_0, \]

we have the following theorem:

**Theorem.** If \( f(t, y), \quad \frac{\partial f}{\partial y}(t, y) \) are continuous and bounded on a rectangle \( (\alpha < t < \beta, \quad a < y < b) \) containing \( (t_0, y_0) \), then there exists an open interval around \( t_0 \), contained in \( (\alpha, \beta) \), where the solution exists and is unique.
We note that the statement of this theorem is not as strong as the one for linear equation.

Below we give two counter examples.

**Example 1.** Loss of uniqueness. Consider

\[ \frac{dy}{dt} = f(t, y) = -\frac{t}{y}, \quad y(-2) = 0. \]

We first note that at \( y = 0 \), which is the initial value of \( y \), we have \( y' = f(t, y) \to \infty \). So the conditions of the Theorem are not satisfied, and we expect something to go wrong.

Solve the equation as a separable equation, we get

\[
\int y \, dy = -\int t \, dt, \quad y^2 + t^2 = c,
\]

and by IC we get \( c = (-2)^2 + 0 = 4 \), so \( y^2 + t^2 = 4 \), and \( y = \pm \sqrt{4 - t^2} \). Both are solutions. We lose uniqueness of solutions.

**Example 2.** Blow-up of solution. Consider a simple non-linear equation:

\[ y' = y^2, \quad y(0) = 1. \]

Note that \( f(t, y) = y^2 \), which is defined for all \( t \) and \( y \). But, due to the non-linearity of \( f \), solution can not be defined for all \( t \).

This equation can be easily solved as a separable equation.

\[
\int \frac{1}{y^2} \, dy = \int dt, \quad -\frac{1}{y} = t + c, \quad y(t) = \frac{-1}{t + c}.
\]

By IC \( y(0) = 1 \), we get \( 1 = -1/(0 + c) \), and so \( c = -1 \), and

\[ y(t) = \frac{-1}{t - 1}. \]

We see that the solution **blows up** as \( t \to 1 \), and can not be defined beyond that point.

This kind of blow-up phenomenon is well-known for nonlinear equations.
2.3: Modeling with first order equations

General modeling concept: derivatives describe “rates of change”.

**Model I**: Exponential growth/decay.

\[ Q(t) = \text{amount of quantity at time } t \]

Assume the rate of change of \( Q(t) \) is proportional to the quantity at time \( t \).

We can write

\[ \frac{dQ}{dt}(t) = r \cdot Q(t), \quad r : \text{rate of growth/decay} \]

If \( r > 0 \): exponential growth
If \( r < 0 \): exponential decay

Differential equation:

\[ Q' = rQ, \quad Q(0) = Q_0. \]

Solve it: separable equation.

\[ \int \frac{1}{Q} dQ = \int r \, dt, \quad \Rightarrow \quad \ln Q = rt + c, \quad \Rightarrow \quad Q(t) = e^{rt+c} = ce^{rt} \]

Here \( r \) is called the *growth rate*. By IC, we get \( Q(0) = C = Q_0 \). The solution is

\[ Q(t) = Q_0e^{rt}. \]

Two concepts:

- **Doubling time** \( T_D \) (only if \( r > 0 \)): is the time that \( Q(T_D) = 2Q_0 \).

\[ Q(T_D) = Q_0e^{rT_D} = 2Q_0, \quad e^{rT_D} = 2, \quad rT_D = \ln 2, \quad T_D = \frac{\ln 2}{r}. \]

- **Half life** (or half time) \( T_H \) (only for \( r < 0 \)): is the time that \( Q(T_H) = \frac{1}{2}Q_0 \).

\[ Q(T_H) = Q_0e^{rT_H} = \frac{1}{2}Q_0, \quad e^{rT_D} = \frac{1}{2}, \quad rT_D = \ln \frac{1}{2} = -\ln 2, \quad T_D = \frac{\ln 2}{-r}. \]

Note here that \( T_H > 0 \) since \( r < 0 \).
NB! \( T_D, T_H \) do not depend on \( Q_0 \). They only depend on \( r \).

**Example 1.** If interest rate is 8\%, compounded continuously, find doubling time.

**Answer.** Since \( r = 0.08 \), we have \( T_D = \frac{\ln 2}{0.08} \).

**Example 2.** A radio active material is reduced to 1/3 after 10 years. Find its half life.

**Answer.** Model: \( \frac{dQ}{dt} = rQ \), \( r \) is rate which is unknown. We have the solution \( Q(t) = Q_0 e^{rt} \). So

\[
Q(10) = \frac{1}{3}Q_0, \quad Q_0 e^{10r} = \frac{1}{3}Q_0, \quad r = -\frac{\ln 3}{10}.
\]

To find the half life, we only need the rate \( r \)

\[
T_H = -\frac{\ln 2}{r} = -\ln 2 \frac{10}{-\ln 3} = 10 \frac{\ln 2}{\ln 3}.
\]

**Model II:** Interest rate/mortgage problems.

**Example 3.** Start an IRA account at age 25. Suppose deposit $2000 at the beginning and $2000 each year after. Interest rate 8\% annually, but assume compounded continuously. Find total amount after 40 years.

**Answer.** Set up the model: Let \( S(t) \) be the amount of money after \( t \) years

\[
\frac{ds}{dt} = 0.08s + 2000, \quad S(0) = 2000.
\]

This is a first order linear equation. Solve it by integrating factor

\[
S' - 0.08S = 2000, \quad \mu = e^{-0.08t}
\]

\[
S(t) = e^{0.08t} \int 2000 \cdot e^{-0.08t} dt = e^{0.08t} \left[ \frac{2000 e^{-0.08t}}{-0.08} + c \right] = \frac{2000}{-0.08} + ce^{0.08t}
\]

By IC,

\[
S(0) = \frac{2000}{-0.08} + c = 2000, \quad C = 2000(1 + \frac{1}{0.08}) = 27000,
\]
we get
\[ S(t) = 27000e^{0.08t} - 25000. \]
When \( t = 40 \), we have
\[ S(40) = 27000 \cdot e^{3.2} - 25000 \approx 637,378. \]
Compare this to the total amount invested: \( 2000 + 2000 \times 40 = 82,000 \).

**Example 4:** A home-buyer can pay $800 per month on mortgage payment. Interest rate is 9% annually, (but compounded continuously), mortgage term is 20 years. Determine maximum amount this buyer can afford to borrow.

**Answer.** Set up the model: Let \( Q(t) \) be the amount borrowed (principle) after \( t \) years
\[
\frac{dQ}{dt} = 0.09Q(t) - 800 \times 12
\]
The terminal condition is given \( Q(20) = 0 \). We must find \( Q(0) \).
Solve the differential equation:
\[
Q' - 0.09Q = -9600, \quad \mu = e^{-0.09t}
\]
\[
Q(t) = e^{0.09t} \int (-9600)e^{-0.09t} dt = e^{0.09t} \left[ -9600 \frac{e^{-0.09t}}{-0.09} + c \right] = \frac{9600}{0.09} + ce^{0.09t}
\]
By terminal condition
\[
Q(20) = \frac{9600}{0.09} + ce^{0.09 \times 20} = 0, \quad c = -\frac{9600}{0.09 \cdot e^{1.8}}
\]
so we get
\[
Q(t) = \frac{9600}{0.09} - \frac{9600}{0.09 \cdot e^{1.8}}e^{0.09t}.
\]
Now we can get the initial amount
\[
Q(0) = \frac{9600}{0.09} - \frac{9600}{0.09 \cdot e^{1.8}} = \frac{9600}{0.09} (1 - e^{-1.8}) \approx 89,034.79.
\]

**Model III:** Mixing Problem.

**Example 5.** At \( t = 0 \), a tank contains \( Q_0 \) lb of salt dissolved in 100 gal of water. Assume that water containing 1/4 lb of salt per gal is entering the tank at a rate of \( r \) gal/min. At the same time, the well-mixed mixture is draining from the tank at the same rate.
(1). Find the amount of salt in the tank at any time \( t \geq 0 \).

(2). When \( t \to \infty \), meaning after a long time, what is the limit amount \( Q_L \)?

**Answer.** Set up the model:

\( Q(t) = \) amount (lb) of salt in the tank at time \( t \) (min)

In-rate: \( r \) gal/min \( \times \) 1/4 lb/gal = \( \frac{r}{4} \) lb/min

Out-rate: \( r \) gal/min \( \times \) \( \frac{Q(t)}{100} \) lb/gal = \( \frac{Q}{100}r \) lb/min

\[
\frac{dQ}{dt} = \text{[In-rate]} - \text{[Out-rate]} = \frac{r}{4} - \frac{r}{100}Q, \quad \text{IC.} \quad Q(0) = Q_0.
\]

(1). Solve the equation

\[
Q' + \frac{r}{100}Q = \frac{r}{4}, \quad \mu = e^{(r/100)t}.
\]

\[
Q(t) = e^{-(r/100)t} \int \frac{r}{4} e^{(r/100)t} dt = e^{-(r/100)t} \left[ \frac{r}{4} e^{(r/100)t} \frac{100}{r} \right] + c = 25 + ce^{-(r/100)t}.
\]

By IC

\[
Q(0) = 25 + c = Q_0, \quad c = Q_0 - 25,
\]

we get

\[
Q(t) = 25 + (Q_0 - 25)e^{-(r/100)t}.
\]

(2). As \( t \to \infty \), the exponential term goes to 0, and we have

\[
Q_L = \lim_{t \to \infty} Q(t) = 25 \text{lb}.
\]

**Example 6.** Tank contains 50 lb of salt dissolved in 100 gal of water. Tank capacity is 400 gal. From \( t = 0 \), 1/4 lb of salt/gal is entering at a rate of 4 gal/min, and the well-mixed mixture is drained at 2 gal/min. Find:

(1) time \( t \) when it overflows;

(2) amount of salt before overflow;

(3) the concentration of salt at overflow.
**Answer.** (1). Since the inflow rate 4 gal/min is larger than the outflow rate 2 gal/min, the tank will be filled up at $t_f$:

$$t_f = \frac{400 - 100}{4 - 2} = 150\text{min}.$$  

(2). Let $Q(t)$ be the amount of salt at $t$ min.

In-rate: $\frac{1}{4} \text{ lb/gal} \times 4 \text{ gal/min} = 1 \text{ lb/min}$

Out-rate: $2 \text{ gal/min} \times \frac{Q(t)}{100 + 2t} \text{ lb/gal} = \frac{Q(t)}{50 + t} \text{ lb/min}$

$$\frac{dQ}{dt} = 1 - \frac{Q}{50 + t}, \quad Q' + \frac{1}{50 + t}Q = 1, \quad Q(0) = 50$$

$$\mu = \exp\left(\int \frac{1}{50 + t} dt\right) = \exp(\ln(50 + t)) = 50 + t$$

$$Q(t) = \frac{1}{50 + t} \int (50 + t) dt = \frac{1}{50 + t}[50t + \frac{1}{2}t^2 + c]$$

By IC:

$$Q(0) = c/50 = 50, \quad c = 2500,$$

We get

$$Q(t) = \frac{50t + t^2/2 + 2500}{50 + t}.$$  

(3). The concentration of salt at overflow time $t = 150$ is

$$\frac{Q(150)}{400} = \frac{50 \cdot 150 + 150^2/2 + 2500}{400(50 + 150)} = \frac{17}{64} \text{ lb/gal}.$$  

**Model IV: Air resistance**

**Example 7.** A ball with mass 0.5 kg is thrown upward with initial velocity 10 m/sec from the roof of a building 30 meter high. Assume air resistance is $|v|/20$. Find the max height above ground the ball reaches.

**Answer.** Let $S(t)$ be the position (m) of the ball at time $t$ sec. Then, the velocity is $v(t) = dS/dt$, and the acceleration is $a = dv/dt$. Let upward be the positive direction. We have by Newton’s Law:

$$F = ma = -mg - \frac{v}{20}, \quad a = -g - \frac{v}{20m} = \frac{dv}{dt}$$
Here $g = 9.8$ is the gravity, and $m = 0.5$ is the mass. We have an equation for $v$:

$$\frac{dv}{dt} = -\frac{1}{10}v - 9.8 = -0.1(v + 98),$$

so

$$\int \frac{1}{v + 98} dv = \int (-0.1) dt, \quad \Rightarrow \quad \ln |v + 98| = -0.1t + c$$

which gives

$$v + 98 = c e^{-0.1t}, \quad \Rightarrow \quad v = -98 + ce^{-0.1t}.$$  

By IC:

$$v(0) = -98 + c = 10, \quad c = 108, \quad \Rightarrow \quad v = -98 + 108e^{-0.1t}.$$  

To find the position $S$, we use $S' = v$ and integrate

$$S(t) = \int v(t) \, dt = \int (-98 + 108e^{-0.1t}) \, dt = -98t + 108e^{-0.1t}/(-0.1) + c$$

By IC for $S$,

$$S(0) = -1080 + c = 30, \quad c = 1110, \quad S(t) = -98t - 1080e^{-0.1t} + 1110.$$  

At the maximum height, we have $v = 0$. Let’s find out the time $T$ when max height is reached.

$$v(T) = 0, \quad -98 + 108e^{-0.1T} = 0, \quad 98 = 108e^{-0.1T}, \quad e^{-0.1T} = 98/108,$$

$$-0.1T = \ln(98/108), \quad T = -\frac{10 \ln(98/108)}{\ln(108/98)} = \ln(108/98).$$  

So the max height $S_M$ is

$$S_M = S(T) = \left( -980 \ln \frac{108}{98} - 1080e^{-0.1\ln(108/98)} \right) + 1110 
= -980 \ln \frac{108}{98} - 1080(98/108) + 1110 \approx 34.78 \text{ m}.$$  

Other possible questions:

- Find the time when the ball hit the ground.
  Solution: Find the time $t = t_H$ for $S(t_H) = 0$.  

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• Find the speed when the ball hit the ground.
  Solution: Compute $|v(t_H)|$.

• Find the total distance traveled by the ball when it hits the ground.
  Solution: Add up twice the max height $S_M$ with the height of the building.
2.5: Autonomous equations and population dynamics

Definition: An autonomous equation is of the form $y' = f(y)$, where the function $f$ for the derivative depends only on $y$, not on $t$.

Simplest example: $y' = ry$, exponential growth/decay, where solution is $y = y_0 e^{rt}$.

Definition: Zeros of $f$ where $f(y) = 0$ are called critical points or equilibrium points, or equilibrium solutions.

Why? Because if $f(y_0) = 0$, then $y(t) = y_0$ is a constant solution. It is called an equilibrium.

Question: Is an equilibrium stable or unstable?

Example 1. $y' = y(y - 2)$. We have two critical points: $y_1 = 0$, $y_2 = 2$.

We see that $y_1 = 0$ is stable, and $y_2 = 2$ is unstable.

Example 2. For the equation $y' = f(y)$ where $f(y)$ is given in the following plot:
• (A). What are the critical points?

• (B). Are they stable or unstable?

• (C) Sketch the solutions in the $t - y$ plan, and describe the behavior of $y$ as $t \to \infty$ (as it depends on the initial value $y(0)$.)

**Answer.** (A). There are three critical points: $y_1 = 1$, $y_2 = 3$, $y_3 = 5$.

(B). To see the stability, we add arrows on the y-axis:

![Diagram showing critical points and arrows indicating stability](image)

We see that $y_1 = 1$ is stable, $y_2 = 3$ is unstable, and $y_3 = 5$ is stable.
Asymptotic behavior for $y$ as $t \to \infty$ depends on the initial value of $y$:

- If $y(0) < 1$, then $y(t) \to 1$,
- If $y(0) = 1$, then $y(t) = 1$;
- If $1 < y(0) < 3$, then $y(t) \to 1$;
- If $y(0) = 3$, then $y(t) = 3$;
- If $3 < y(0) < 5$, then $y(t) \to 5$;
- If $y(0) = 5$, then $y(t) = 5$;
- if $y(t) > 5$, then $y(t) \to 5$.

Stability: is not only stable or unstable.

**Example 3.** For $y' = y^2$, we have only one critical point $y_1 = 0$. For $y < 0$, we have $y' > 0$, and for $y > 0$ we also have $y' > 0$. So solution is increasing
on both intervals. So on the interval \( y < 0 \), solution approaches \( y = 0 \) as \( t \) grows, so it is stable. But on the interval \( y > 0 \), solution grows and leaves \( y = 0 \), and it is unstable. This type of critical point is called \textit{semi-stable}. This happens when one has a double root for \( f(y) = 0 \).

Example 4. For equation \( y' = f(y) \) where \( f(y) \) is given in the plot

\begin{itemize}
\item (A). Identify equilibrium points;
\item (B). Discuss their stabilities;
\item (C). Sketch solution in \( y - t \) plan;
\item (D). Discuss asymptotic behavior as \( t \to \infty \).
\end{itemize}

\textbf{Answer.} (A). \( y = 0, y = 1, y = 2, y = 3 \) are the critical points.
(B). \( y = 0 \) is stable, \( y = 1 \) is semi-stable, \( y = 2 \) is unstable, and \( y = 3 \) is stable.
(C). The Sketch is given in the plot:
(D). The asymptotic behavior as \( t \to \infty \) depends on the initial data.

- If \( y(0) < 1 \), then \( y \to 0 \);
- If \( 1 \leq y(0) < 2 \), then \( y \to 1 \);
- If \( y(0) = 2 \), then \( y(t) = 2 \);
- If \( y(0) > 2 \), then \( y \to 3 \).

Application in population dynamics: let \( y(t) \) be the population of a species.

\[
\frac{dy}{dt} = (r - ay)y. \quad \text{the logistic equation}
\]

\[
\frac{dy}{dt} = r(1 - \frac{y}{k})y, \quad k = \frac{r}{a},
\]

\( r \) = intrinsic growth rate,
\( k \) = environmental carrying capacity.

Critical points: \( y = 0, \ y = k \). Here \( y = 0 \) is unstable, and \( y = k \) is stable.
If \( 0 < y(0) < k \), then \( y \to k \) as \( t \) grows.