The gradient vector is \( \nabla z(x, y) = \langle -6x, -2y \rangle \); at the given point, it is \( \nabla z(0, 1) = \langle 0, -2 \rangle \).

(a) The slope in the direction of \( u = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle \) is the directional derivative in that direction. We have already been given a unit vector, so we don’t need to rescale. Just compute,

\[
D_u f(0, 1) = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle \cdot \langle 0, -2 \rangle = -\sqrt{2}.
\]

So the slope in the given direction is \(-\sqrt{2}\).

(b) In order to stay at constant elevation, you would need to walk along a level curve—in other words, orthogonally to the gradient \( \nabla z(0, 1) = \langle 0, -2 \rangle \). So you would walk in the direction of \( \langle 1, 0 \rangle \) (or in the opposite direction).

(c) Steepest ascent occurs in the direction of the gradient vector \( \langle 0, -2 \rangle \), and the maximal slope is equal to the length of this vector, i.e. 2.

2 We have \( f_x = y^2 + \frac{1}{2}x - 4 \), \( f_y = 2xy \). Critical points occur when \( f_x = f_y = 0 \), i.e.

\[
y^2 = 4 - \frac{1}{2}x, \quad 2xy = 0.
\]

The second equation implies that either \( x = 0 \) or \( y = 0 \). If \( x = 0 \), then the first equation implies that \( y = \pm 2 \). So \( (0, \pm 2) \) are two critical points. If, on the other hand, \( y = 0 \), then the first equation says that \( x = 8 \), so \((8, 0)\) is the only other critical point.

Now \( f_{xx} = 1/2, f_{yy} = 2x \), and \( f_{xy} = 2y \). For each critical point, we calculate \( D = f_{xx}f_{yy} - f_{xy}^2 = x - 4y^2 \):

- \((0, 2)\): \( D < 0 \), so \( f \) has a saddle point at \((0, 2)\).
- \((0, -2)\): \( D < 0 \), so \( f \) has a saddle point at \((0, -2)\).
- \((8, 0)\): \( D > 0 \), and \( f_{xx} = 1/2 > 0 \), so \( f \) has a local minimum at \((8, 0)\).

3 If \( x^2 + y^2 = 1 \), then \( x^2 = 1 - y^2 \), and consequently we have \( f(x, y) = y(1 - y^2) \). This is a function of just one variable; we apply the methods of single-variable calculus. The first derivative is \( 1 - 3y^2 \), which has zeros at \( y = \pm 1/\sqrt{3} \). Putting this into \( x^2 + y^2 = 1 \) gives us the four points \((\pm \sqrt{2/3}, 1/\sqrt{3}), (\pm \sqrt{2/3}, -1/\sqrt{3})\). At the former points, the value of \( f \) is \( 2/(3\sqrt{3}) \); at the latter, it is \(-2/(3\sqrt{3})\). These are, respectively, the maximum and minimum values of \( f \) subject to the given constraint.

4 Consider the graph \( z = 2 - \sqrt{4 - x^2 - y^2} \). This is the lower hemisphere of the sphere with centre \((0, 0, 2)\) and radius 2 (Why? Rearrange the equation to get \( z - 2 = -\sqrt{4 - x^2 - y^2} \). This says that \( z - 2 < 0 \). Keeping this in mind, square both sides and rearrange to get \( x^2 + y^2 + (z - 2)^2 = 4 \). This is the equation of the sphere centre \((0, 0, 2)\) radius 2; the fact that \( z - 2 < 0 \) tells us to consider only the lower hemisphere). The region of integration is the disk in the \( xy \)-plane with centre \((0, 0)\) and radius 2, which is precisely the projection (or “shadow”) of this hemisphere on the \( xy \)-plane.

Now, the integral of a positive function over a region \( D \) is equal to the volume of the solid with base \( D \), bounded above by the graph of that function. By drawing a picture, we see that this volume is equal to the volume of the cylinder (radius 2, height 2), minus the volume of the hemisphere (radius 2). So we get

\[
\iiint_D \left( 2 - \sqrt{4 - x^2 - y^2} \right) \, dA = 2\pi(2^2) - \frac{1}{2}\frac{4}{3}\pi(2^3) = \frac{8\pi}{3}.
\]

5 The region of integration (call it \( D \)) is bounded on the left (and above) by the parabola \( x = y^2 \) (a.k.a. \( x = y^{1/2} \)), on the right by the line \( x = 4 \), and below by the line \( y = 0 \). To change the order of
integration, draw a picture of \( D \) and discover that for each \( x, y \) is bounded below by 0 and above by the curve \( y = x^{1/2} \). Since \( x \) ranges from 0 to 4, the integral becomes

\[
\int_0^4 \int_0^{x^{1/2}} ye^x dy dx.
\]

After evaluating the inner integral and substituting the limits of integration, we get the integral \( \int_0^4 \frac{1}{2} e^{x^2} dx \). After a substitution \( u = x^2 \), this becomes \( \frac{1}{4} \int_0^{16} e^u du \), which equals \( \frac{1}{4}(e^{16} - 1) \).

6 The region of integration \( R \) is a sector of an annulus. This region is much better-described in polar coordinates than in rectangular coordinates; its description in polar coordinates is

\[
R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi/4\}.
\]

Using the identity \( x^2 + y^2 = r^2 \), the function we’re integrating becomes \( r^{-3} \). Finally, replacing \( dy dx \) by \( r dr d\theta \) (don’t forget the \( r \)!), we get the integral

\[
\int_0^{\pi/4} \int_1^2 r^{-2} dr d\theta = \int_0^{\pi/4} \frac{1}{2} d\theta = \frac{\pi}{8}.
\]

7 To find the area of a surface which doesn’t lie in the \( xy \)-plane, we use the surface area formula. To apply this formula, we need to know \( D \) (the shadow in the \( xy \)-plane), and \( f \) (the function whose graph is the surface in question). Here we have \( f(x, y) = 4 - \sqrt{x^2 + y^2} \). For \( D \), draw the two cones, and find that they intersect in a circle, centre \((0, 0, 2)\), radius \( 2 \). So the region \( D \) is the disk in the \( xy \)-plane, centre \((0, 0)\), radius \( 2 \).

Now, to apply the surface area formula, calculate the first partial derivatives of \( f 

\[
f_x(x, y) = -x(x^2 + y^2)^{-\frac{1}{2}}, \quad f_y(x, y) = -y(x^2 + y^2)^{-\frac{1}{2}}.
\]

Now the surface area formula tells us that the area of the surface is equal to

\[
\iint_D \sqrt{x^2 + y^2 + 1} dA = \sqrt{2} \iint_D dA = \sqrt{2} \cdot \text{Area}(D) = 4\pi\sqrt{2}.
\]

8 Recall that the average value of a function \( f \) over a region \( E \) is equal to \( \frac{1}{\text{Volume}(E)} \iiint_E f(x, y, z) dV \). Here, \( E \) is a rectangular prism, with volume \( 2 \cdot 4 \cdot 1 = 8\pi \). So the average value of \( f \) over \( E \) is

\[
\frac{1}{8\pi} \int_0^1 \int_0^4 \int_0^{2\pi} y \sin x + z dx dy dz
\]

\[
= \frac{1}{8\pi} \int_0^1 \int_0^4 2\pi z dy dz
\]

\[
= \int_0^1 z dz = \frac{1}{2}.
\]

9 Draw a picture of the region. From the picture, discover that in spherical coordinates this region is

\[
E = \{(\rho, \theta, \varphi) \mid 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}\}.
\]

The volume of any region \( E \) is given by the triple integral \( \iiint_E dV \). Changing to spherical coordinates (remembering that \( dV = \rho^2 \sin \varphi d\rho d\theta d\varphi \)), we get

\[
\int_{\frac{\pi}{4}}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin \varphi d\rho d\theta d\varphi.
\]
10 (a) Add the two formulae together and divide by 2 to get \( x = \frac{1}{2}(u + v) \). Then subtract the second from the first and divide by 2 to get \( y = \frac{1}{2}(u - v) \).

(b) Find the four first partial derivatives: \( x_u = \frac{1}{2}, \ x_v = \frac{1}{2}, \ y_u = \frac{1}{2}, \ y_v = -\frac{1}{2} \). Now plug these in to the formula for the Jacobian (i.e. take the determinant of the Jacobi matrix) to find that \( \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2} \).

(c) We need to convert the region, the function and the differential into the coordinates \((u, v)\). In this question it is easy to deal with the region \( D \), since the description we’re given tells us immediately that \( 1 \leq v \leq 2 \) and \( 0 \leq u \leq 3 \). To translate the function \( x^2 - y^2 \) into the coordinates \((u, v)\), just notice that \( x^2 - y^2 = (x + y)(x - y) = uv \) (or, if you didn’t notice this, substitute the formulas you found in part (a) for \( x \) and \( y \), and simplify). As for the differential, we replace \( dA \) by \( \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \ du \ dv = \frac{1}{2} du \ dv \). Combining the three ingredients, we see that the integral in question is equal to

\[
\int_{1}^{2} \int_{0}^{3} \frac{1}{2} uv \ du \ dv = \frac{1}{4} \int_{1}^{2} 9v \ dv = \frac{27}{8}.
\]