MASS (FALL 07): TOPICS IN PROBABILITY
ASSIGNMENT 9

Submit on Wednesday, 11/14. Prove all your statements. *You should be able to do problems 1,3 on your own.*

Throughout this assignment, X_i is a *stationary* sequence of (measurable) random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. $\mathscr{B}(\mathbb{R})$ is the Borel sigma–algebra of $\mathbb{R}$, and $\mathbb{R}^{\mathbb{N}} := \{(x_1, x_2, \ldots) : x_i \in \mathbb{R} \ \ (i \geq 1)\}$. A *cylinder* is a subset of $\mathbb{R}^{\mathbb{N}}$ of the form $[B_1, \ldots, B_n] := \{\underline{x} \in \mathbb{R}^{\mathbb{N}} : x_1 \in B_1, \ldots, x_n \in B_n\}$ where $B_i \in \mathscr{B}(\mathbb{R})$. The *Borel sigma algebra* of $\mathbb{R}^{\mathbb{N}}$ is the sigma algebra generated by the cylinders.

(1) Show:
   (a) If X_1 is a simple function, then X_i are simple functions for all i, and $\mathbb{E}(X_i) = \mathbb{E}(X_1)$ for all i.
   (b) If X_1 is absolutely integrable, then X_i are absolutely integrable and $\mathbb{E}(X_i) = \mathbb{E}(X_1)$ for all i.

(2) Let $\mathscr{A}$ denote the collection of all finite disjoint unions of cylinders in $\mathbb{R}^{\mathbb{N}}$. Prove that $\mathscr{A}$ is an algebra (i.e. $\mathscr{A}$ is closed under finite intersections, complements, and finite unions).

(3) Prove that for every $E \in \mathscr{B}(\mathbb{R}^{\mathbb{N}})$
   (a) $[(X_1, X_2, X_3, \ldots) \in E] := \{\omega \in \Omega : (X_1(\omega), X_2(\omega), X_3(\omega), \ldots) \in E\}$ is $\mathscr{F}$–measurable;
   (b) $\mathbb{P}[(X_1, X_2, X_3, \ldots) \in E] = \mathbb{P}[(X_{1+k}, X_{2+k}, X_{3+k}, \ldots) \in E] \quad (k \geq 1)$

   Guidance: Let $\mathscr{M}$ denote the collection of all sets $E \in \mathscr{B}(\mathbb{R}^N)$ with properties (a),(b) above. Use the monotone class theorem (assign.3) to show that $\mathscr{M} = \mathscr{B}(\mathbb{R}^{\mathbb{N}})$.

(4) Suppose $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is $\mathscr{B}(\mathbb{R}^{\mathbb{N}})$–measurable, and define

$$Y_k(\omega) := F\big(X_k(\omega), X_{k+1}(\omega), X_{k+2}(\omega), \ldots\big) \qquad (k \geq 1).$$

   Prove that Y_k is a stationary sequence of random variables. Deduce that τ_k from the proof of the ergodic theorem form a stationary sequence.