MASS (FALL 07): TOPICS IN PROBABILITY
ASSIGNMENT 8

Submit on Wednesday, 11/7. Prove all your statements.

Throughout this assignment, $(\Omega, \mathscr{F}, \mathbb{P})$ is a fixed probability space. *You should be able to do problems 1 (a), 2 (a) on your own.*

(1) Suppose X is a bounded random variable, and define $c(t) := \ln \mathbb{E}(e^{tX})$, $I(\alpha) := \sup_{t\in\mathbb{R}}\{t\alpha - c(t)\}$. Prove:
 (a) $I(\mathbb{E}(X)) = 0$;
 (b) $\alpha \neq \mathbb{E}(X) \Rightarrow I(\alpha) > 0$.

(2) Suppose X_i are bounded iid's distributed like X, and set $c(t) = \ln \mathbb{E}(e^{tX})$ and $I(\alpha) := \sup_{t\in\mathbb{R}}\{t\alpha - c(t)\}$.
 (a) Use the 'exponential Chebyshev trick' to prove

$$(*) \qquad \limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{P}[\frac{1}{n}(X_1 + \cdots + X_n) \in F] \leq -\inf_{\alpha\in F} I(\alpha).$$

 for sets of the form $F = [\alpha, \infty), (-\infty, \alpha]$
 (b) Prove that for any two sequences $a_n, b_n \geq 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \ln[a_n + b_n] = \max\left\{\limsup_{n\to\infty} \frac{1}{n} \ln[a_n], \limsup_{n\to\infty} \frac{1}{n} \ln[b_n]\right\}$$

 (c) Prove $(*)$ for sets of the form $F = (-\infty, \alpha] \cup [\beta, \infty)$.
 (d) Prove $(*)$ for every closed set F.