MASS (FALL 07): TOPICS IN PROBABILITY
ASSIGNMENT 6

Submit on Wednesday, 10/24. Prove all your statements.

Throughout this assignment, $(\Omega, \mathscr{F}, \mathbb{P})$ is a fixed probability space, and X, Y are measurable random variables. *Please try to do problems 1,2 on your own.*

(1) The *variance* of an integrable measurable random variable X is the (possibly infinite) quantity $\mathrm{Var}(X) := \mathbb{E}[(X - \mathbb{E}(X))^2]$. Prove:
   (a) $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$.
   (b) $\mathrm{Var}(X) < \infty \Rightarrow \mathbb{E}(|X|) < \infty$, but $\mathbb{E}(|X|) < \infty \not\Rightarrow \mathrm{Var}(X) < \infty$.
   (c) If $\mathrm{Var}(X) = 0$, then $X = $ *const* with full probability, i.e. $\exists c$ s.t. $\mathbb{P}[X \neq c] = 0$.
   (d) If X, Y are *independent*, then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

(2) *Infinite Bernoulli trials:* Construct a probability space which models the results of infinite tosses of a fair coin, assuming that the tosses are independent. Write down the set corresponding to the event: "the total number of heads at the first n tosses, divided by n, converges to $1/2$ as $n \to \infty$". Is this set measurable?

(3) Consider the unit interval $[0, 1]$ equipped with the uniform distribution (Lebesgue measure). Call $\omega \in [0, 1]$ *dyadic*, if it is of the form $\omega = m/2^n$ with $m, n \in \mathbb{N} \cup \{0\}$, and let $D := \{x \in [0, 1] : x \text{ is dyadic}\}$. Every $x \in [0, 1] \setminus D$ has a unique binary expansion

$$\omega = 0.\omega_1\omega_2\omega_3\cdots, \text{ equivalently } \omega = \sum_{i=1}^{\infty} \frac{\omega_i}{2^i} \quad (\omega_i = 0, 1).$$

Define $X_i : [0, 1] \to \{0, 1\}$ by $X_i(\omega) = \begin{cases} \omega_i & \omega \in [0, 1] \setminus D \\ \text{who cares?} & \omega \in D = \text{set of probability zero.} \end{cases}$

Prove: $X_1, X_2, X_3, \ldots$ are mutually independent random variables.

(4) Two probability spaces $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1)$,$(\Omega_2, \mathscr{F}_2, \mathbb{P}_2)$ are called *isomorphic* if $\exists \Omega_1' \in \mathscr{F}_1$, $\Omega_2' \in \mathscr{F}_2$ and an invertible map $\pi : \Omega_1' \to \Omega_2'$ such that
   - $\mathbb{P}_1(\Omega_1 \setminus \Omega_1') = 0 = \mathbb{P}_2(\Omega_2 \setminus \Omega_2')$
   - π is measurable, i.e. $E \in \Omega_2', E \in \mathscr{F}_2 \Rightarrow \pi^{-1}(E) \in \mathscr{F}_1$
   - $\mathbb{P}_1 \circ \pi^{-1} = \mathbb{P}_2$, i.e. $E \in \Omega_2', E \in \mathscr{F}_2 \Rightarrow \mathbb{P}_1(\pi^{-1}(E)) = \mathbb{P}_2(E)$.

   Prove that the unit interval equipped with the uniform distribution is isomorphic to the probability space of infinite, independent, fair coin tosses.