# MASS (FALL 07): TOPICS IN PROBABILITY
## ASSIGNMENT 5

Submit on Wednesday, 10/3. Prove all your statements.

Througout this assignment, $(\Omega, \mathscr{F}, \mu)$ is a probability space, and $X, Y : \Omega \to \mathbb{R}$ are $\mathscr{F}$ measurable. The 'basic properties of the expectation' are: (1) Linearity: $\mathbb{E}(\alpha X + \beta Y) = \alpha\mathbb{E}(X) + \beta\mathbb{E}(Y)$; (2) Monotonicity: $X \leq Y \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y)$; (3) Triangle inequality: $|\mathbb{E}(X)| \leq \mathbb{E}(|X|)$. The notation $a \wedge b$ means $\min\{a, b\}$.

(1) Prove the basic properties of the expectation assuming that $X$, $Y$ are simple.
(2) Prove the basic properties of the expectation assuming that $X, Y$ are bounded.
(3) Suppose $X, Y \geq 0$ and set $Z := X + Y$.
  (a) Prove that $\mathbb{E}(Z) \geq \mathbb{E}(X) + \mathbb{E}(Y)$.
  (b) Define $Z := X + Y$ and suppose $0 \leq Z' \leq Z$ and $Z'$ is bounded. Define $X' := X \wedge Z'$, $Y' := Z' - X'$. Prove that $0 \leq X' \leq X$ and $0 \leq Y' \leq Y$. Deduce that $\mathbb{E}(Z) \leq \mathbb{E}(X) + \mathbb{E}(Y)$.
  (c) Prove the basic properties of the expectation for non-negative $X, Y$.
(4) Suppose $X, Y$ are absolutely integrable.
  (a) Prove that
$$\mathbb{E}[(X+Y)^+] = \mathbb{E}[(X^+ + Y^+)1_{[X^+ + Y^+ > X^- + Y^-]}] - \mathbb{E}[(X^- + Y^-)1_{[X^+ + Y^+ > X^- + Y^-]}].$$
  Find a similar identity for the negative parts. Deduce that $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.
  (b) Prove the basic properties of the expectation for absolutely integrable random variables.
  (c) Show by example that linearity may not hold for *one*–sided integrable random variables.
(5) Suppose $\mathbb{E}(|X|) = 0$. Prove that $\mu\{\omega : X(\omega) \neq 0\} = 0$.
(6) Suppose $X_1, X_2, X_3, \ldots$ are absolutely integrable random variables such that $\sum_{n\geq 1} \mathbb{E}(|X_n|) < \infty$. Prove that
$$\mu\left\{\omega \in \Omega : \sum_{n=1}^{\infty} X_n(\omega) \text{ converges}\right\} = 1.$$
We say that "$\sum_{n\geq 1} X_n(\omega)$ converges with probability one" (or "almost surely").
(7) Suppose $X_1, X_2, X_3, \ldots$ are non-negative, measurable, random variables. Prove that
$$\mathbb{E}(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} \mathbb{E}(X_n).$$
Give an example showing that a strict inequality is possible.