

**MASS (FALL 07): TOPICS IN PROBABILITY
ASSIGNMENT 3**

Submit on Wednesday, 9/17. Prove all your statements.

The purpose of this assignment is twofold: to guide you through the proof of the *Monotone Class Theorem* (a basic tool in Measure Theory), and to train you in the manipulation of closure axioms for collections of sets.

Definition 1. Let Ω be a set. An algebra of subsets of Ω is a collection \mathcal{A} of subsets of Ω such that

- (1) $\emptyset, \Omega \in \mathcal{A}$
- (2) $A_1, \dots, A_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^n A_i, \bigcap_{i=1}^n A_i \in \mathcal{A}$
- (3) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

Definition 2. A sequence of sets $\{E_n\}_{n=1}^\infty$ is called increasing (respectively, decreasing) if for all n , $E_n \subseteq E_{n+1}$ (respectively, $E_n \supseteq E_{n+1}$).

It is customary to write $E_n \uparrow E$, when $\{E_n\}_{n \geq 1}$ is increasing and $E = \bigcup_{n \geq 1} E_n$. Similarly, $E_n \downarrow E$ means that $\{E_n\}_{n \geq 1}$ is decreasing, and $E = \bigcap_{n \geq 1} E_n$.

Definition 3. Let Ω be a set. A monotone class of subsets of Ω is a collection \mathcal{M} of subsets of Ω which contains the empty set, and such that if $E_n \in \mathcal{M}$ for $n = 1, 2, 3, \dots$, and $E_n \uparrow E$ or $E_n \downarrow E$, then $E \in \mathcal{M}$.

Theorem 1 (Monotone Class Theorem). Suppose \mathcal{M} is a monotone class of subsets of Ω . If \mathcal{M} contains an algebra \mathcal{A} , then it contains a σ -algebra which contains \mathcal{A} .

- (1) Show there exists a monotone class $\mathcal{M}(\mathcal{A})$ which contains \mathcal{A} , which is minimal in the sense that it is contained in any other monotone class which contains \mathcal{A} .
 - (2) Prove that $\mathcal{M}(\mathcal{A})$ is closed under taking complements: $A \in \mathcal{M}(\mathcal{A}) \Rightarrow A^c \in \mathcal{M}(\mathcal{A})$. (Hint: Think about $\mathcal{M}_1 := \{E' \in \mathcal{M}(\mathcal{A}) : (E')^c \in \mathcal{M}(\mathcal{A})\}$.)
 - (3) Show that if $E \in \mathcal{M}(\mathcal{A})$ and $A \in \mathcal{A}$, then $E \cup A \in \mathcal{M}(\mathcal{A})$. (Hint: consider the collection of sets $E \in \mathcal{M}$ with this property.)
 - (4) Use the previous step to show that $\mathcal{M}(\mathcal{A})$ is closed under the formation of finite unions.
 - (5) Show that $\mathcal{M}(\mathcal{A})$ is a σ -algebra. This completes the proof.
- * (6) Use the monotone class theorem to prove the following statement, for the uniform distribution on $[0, 1]$:¹ any Borel measurable event which is independent of any interval must have probability zero or one.

¹The statement is true for every probability measure defined on all Borel sets of the unit interval.