MASS (FALL 07): TOPICS IN PROBABILITY
ASSIGNMENT 2

Submit on Wednesday, 9/12. Prove all your statements.

(1) Suppose X, Y are random variables on a discrete probability space, and assume X, Y have finite expectations. Prove: $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

(2) We say that two random variables X, Y are *independent*, if for every $x, y \in \mathbb{R}$ the events $[X(\omega) = x], [Y(\omega) = y]$ are independent. Prove:
   (a) X, Y independent $\Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, assuming all expectations exist.
   (b) $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \not\Rightarrow X, Y$ are independent.

(3) The *expectation* of a probability measure on a set $\mathbb{N} \cup \{0\}$ is defined by $\sum n\mathbb{P}\{n\}$. Prove that the expectation of the Poisson distribution with rate λ is λ.

(4) An infinite collection of points is chosen in random in the plane in such a way that (a) the expected number of points in a square is proportional to the area of the square, and (b) the number of points in non-overlapping squares are mutually independent (Figure 1). Write a probability model for the number of points in the unit square. Justify your choices!

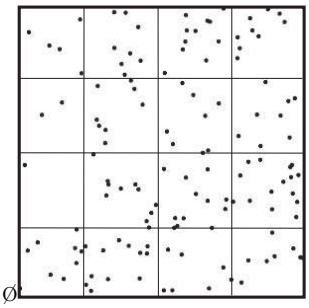

FIGURE 1. A random collection of points in the plane (taken from the web-page of Y. Peres.)

$*$ (5) [*Not for submission, but try to think about this.*] In 1944, the Germans started bombarding London with a new weapon, the "V1" rockets. You are a statistician working for British Military Intelligence, and your boss wants to know if the V1s are aimed at particular targets (perhaps indicating some strategy), or are just shot indiscriminately.

You have the following data at your disposal:[1] A survey of 537 V1 attacks shows that if one divides the area of South London into 576 areas of 0.25 square kilometers each, then

- 229 areas had 0 hits

[1] R.D. Clarke, Journal of the Institute of Actuaries **72**, page 481 (1946). This example, due to Clarke, is mentioned by Feller.

- 211 areas had 1 hit
- 93 areas had 2 hits
- 35 areas had 3 hits
- 7 areas had 4 hits
- 1 area had more than 4 hits

How would you approach this problem? What would you answer your boss?