MASS (FALL 07): TOPICS IN PROBABILITY
ASSIGNMENT 1

Submit on Wednesday, 9/5. Prove all your statements.

(1) Suppose $(\Omega, \mathscr{F}, \mathbb{P})$ is a discrete probability space. Prove:
   (a) $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(\Omega) = 1$;
   (b) *Additivity:* Suppose $E_1, \ldots, E_n$ are pairwise disjoint, namely $i \neq j \Rightarrow E_i \cap E_j = \varnothing$, then $\mathbb{P}[\biguplus_{i=1}^n E_i] = \sum_{i=1}^n \mathbb{P}[E_i]$;
   (c) *Law of total probability:* $\mathbb{P}[E^c] = 1 - \mathbb{P}[E]$;
   (d) *Inclusion–Exclusion:* If $E, F \in \mathscr{F}$, then $\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F]$;
   (e) *Monotonicity:* If $E \subseteq F$, then $\mathbb{P}[E] \leq \mathbb{P}[F]$.

(2) The university computer produces upon request a class list with the birthday date of each student (day, month, but no year).
   (a) Set a probability space which describes the possible lists of dates. Specify your assumptions explicitly.
   (b) In a class of 20 students, what is the probability that at least two have the same birthday? Specify the set corresponding to the event whose probability you are calculating.

(3) An investor invests \$1000 in a stock portfolio, and then does not touch the money for 20 years. Write a probability model for the value of the portfolio after 20 years, assuming
   (a) Every year, the value of the portfolio increases by %15 with probability %80, or decreases by %30 with probability %20.
   (b) Portfolio performance in different years is independent.

(4) A gas station is re-stocked with fuel once a day. Suppose the station serves on average 100 car a day, and that every car takes 10 gallons of fuel. How much fuel should the station have at the beginning of the day to be able to serve all customers with probability %95? %99? %99.9? Guidance:
   (a) Set a probability space model for the number of customers per day, using parameter if needed. Make all your assumptions explicit.
   (b) Estimate the parameter.
   (c) Answer the question.

   [The average of a probability distribution on $\mathbb{N} \cup \{0\}$ is $\sum n\mathbb{P}\{n\}$.]

(5) Suppose $\mathbb{P}(A), \mathbb{P}(B) \neq 0$. The following statements are equivalent:
   (a) A, B are independent;
   (b) $\mathbb{P}(A|B) = \mathbb{P}(A)$;
   (c) $\mathbb{P}(B|A) = \mathbb{P}(B)$.

$*$ (6) Suppose $A_1, \ldots, A_n$ are mutually independent events. Prove that $A_1^c, \ldots, A_n^c$ are mutually independent. Here and throughout $E^c := \Omega \setminus E$.