MASS Fall 2007: Topics in Probability

Fall 2006

Generalities

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Office hours. Mondays, after class or by appointment.

Course hompage. `http://www.math.psu.edu/sarig/MASS07.html`

Assessment. Weekly homework (submit on Wednesdays), written mid–term exam (Thursday, October 4th. 10:00–12:00, 113 McAllister), oral final examination (December 10, 12, 14), and research project.

Mode of instruction: three weekly lectures (MWF 1:25–2:15, 113 McAllister) and one weekly recitation session with the TA, **Arseny Egorov** (R 1:25–2:15, 113 McAllister).

Aim, program, and suggested reading

Randomness, 'noise', and uncertainty are crucial factors in the behavior of many real–life systems, be they subatomic particles or financial markets. As a result, probabilistic techniques are used in almost all scientific disciplines.

To study randomness at a rigorous mathematical level, one needs to confront a variety of challenges, some conceptual (what is randomness?), some technical (how do we analyze randomness?). To address these issues, the modern probabilist uses sophisticated tools from analysis. It is a thing of beauty that in doing so one often gains deep, sometimes unexpected, insights into analysis. Many deep theorems in analysis owe their discovery to 'the probabilistic way of thought'.

The aim of the course is to present probability theory from an analytic point of view, and certain topics in analysis (measure theory, Fourier transform, convexity, fixed point theorems etc) from a probabilistic perspective.

Part 1: Foundations of the mathematical theory of probability

Introduction: How is it possible that Newtonian coin chooses the side it falls on randomly, eventhough its equations of motion are deterministic. Basic framework: Sample space, Measurable Events, Probability.

Discrete probability spaces. Definition. Examples: finite number of Bernoulli trials and the Bernoulli measure, finite step random walk and the Binomial distribution, London blitz and the Poisson distribution.

General probability spaces. Abstract definition. Basic properties (monotonicity, inclusion–exclusion, continuity). Borel-Cantelli Lemma (easy half). Conditional probabilities. Independence. Borel–Cantelli (hard half).

Examples of non–discrete probability spaces. Motivating example: $U[0,1]$. The fundamental difficulty. Carathéodory Extension Theorem (no proof). The Borel σ–algebra and the Lebesgue measure. Distribution functions and the measures they determine (Stieltjes measures). Disasters and the exponential distribution. Density functions. Events of zero probability.

Random variables and expectation. Measurability. Stability under limits. Expectation and variance for discrete RV's. Expectation for general RV's. Limit behavior (Bounded, Monotone, and Dominated Convergence theorems). Modern theory of integration. Expectation and independence. Baby Fubini theorem. Variance. Cauchy–Schwarz.

Part 2: Coin tossing, and the basic theory of iid's

The probability space. Probability model for Bernoulli trials. Probability model for a sequence of iid's. Rademacher and Walsh systems (optional).

Weak Law of Large Numbers. Chebyshev inequality. WLLN. Large deviation upper bounds (bounded r.v.'s). Definition of the Legendre transform and the moment generating function. Calculation for coin tossing. Thermodynamic interpretations. Application to Monte–Carlo simulations. Bernstein's proof of the Weierstrass Approximation Theorem (if there is time).

Strong Law of Large Numbers. SLLN for bounded iid's. If there is time: SLLN for RV's with finite variance (Kolmogoroff's inequality). Normal numbers. Speed of a random walk.

Statement of Central Limit Theorem. Statement of the CLT. The characteristic function. Sketch of the proof of the CLT. Identification of key difficulties: inversion problem, continuity problem.

Convergence in distribution. Definition. Relation to convergence *of* distributions. Tightness. Helly's compactness theorem.

Fourier transforms. Basic properties. Inversion formula. Tightness inequality. Lévy continuity theorem. Proof of CLT.

Part 3: The simple random walk

Definitions. Probability space. Recurrence and transience. Formulation in terms of first return time probabilities. Renewal equation. Criterion for recurrence in terms of return time probabilities.

Local Limit Theorem. De Moivre–Laplace theorem (Stone's proof). Polya's theorem on recurrence of the simple random walk.

Law of the Iterated Logarithm. Statement. If there is time: Feller's proof.

Part 4: Markov Chains

The probability space. Stochastic matrices, probability vectors, irreducibility, aperiodicity, representation by graphs. Time evolution of probability densities.

Examples of Markov Chain Models. Ehrenfest model, Gambler Ruin Problem, Google PageRank algorithm, Branching process, Epidemics model.

Convegence to equilibrium. Exponential convergence to equilibrium for finite state mixing Markov chains (Sinai's proof). Kac formula.

Analysis of the Ehrenfest model. Stationary distribution. Reversibility paradox and its resolution.

Texts. The following are all excellent (but you do not need to buy them):

(1) L. Breiman: *Probability.* Classics in Applied Mathematics, SIAM.

Analytic in taste. Excellent account of limit theorems. Requires solid background in real analysis as provided, e.g., by Royden's book (see below).

(2) W. Feller: *An introduction to Probability Theory and its Applications.* 2 volumes. Wiley International.

Volume 1: Combinatorial in taste. Arguably the best available account of probability theory on discrete probability spaces. Rich in examples.

Volume 2: Encyclopædic account of probability theory on non-discrete probability spaces. Good for reference, but too overwhelming for self–study.

(3) Ya. G. Sinai: *Probability Theory, An Introductory Course.* Springer–Verlag.

Excellent concise overview of the subject. Good for self study, but not for reference.

(4) H.L. Royden: *Real analysis.* McGraw–Hill.

Very readable text on real analysis and measure theory (no probability though!)