MATH 231H SOLUTIONS TO SPECIAL ASSIGNMENT FIVE

Problem 0.

(a) $d(u+v) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u+v)dx_i = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u+v)dx_i = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}dx_i + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}dx_i = du + dv$.

(b) $d(uv) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(uv)dx_i = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(uv)dx_i = \sum_{i=1}^{n} v\frac{\partial u}{\partial x_i}dx_i + \sum_{i=1}^{n} u\frac{\partial v}{\partial x_i}dx_i = vdu + udv$

Problem 1.

(a) $u(\mathbf{x}) = (x^2+y^2+z^2)^{-1/2}$ so $u_x = -x(x^2+y^2+z^2)^{-3/2}$, whence

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2} = \frac{3x^2-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}.$$

By symmetry $u_{yy} = \frac{3y^2-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}$, and $u_{zz} = \frac{3z^2-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}$ and we see that $\Delta u = u_{xx}+u_{yy}+u_{zz} = 0$.

(b) $u(\mathbf{x}) = e^{\mathbf{a}\cdot\mathbf{x}}$ so $u_x = a_1 e^{\mathbf{a}\cdot\mathbf{x}}$ and $u_{xx} = a_1^2 e^{\mathbf{a}\cdot\mathbf{x}} = a_1^2 u(\mathbf{x})$. By symmetry, $u_{yy} = a_2^2 u(\mathbf{y})$ and $u_{zz} = a_3^2 u(\mathbf{z})$. Adding them all up gives $\Delta u = |\mathbf{a}|^2 u$.

Problem 2.

(1) The notation reflect what you get by treating ∇ as a vector of numbers and using the ordinary rules of scalar, dot, and vector product.

(2) $\text{curl}(\text{grad}\, u) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} = (\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial z \partial y})\mathbf{i} - (\frac{\partial^2 u}{\partial z \partial x} - \frac{\partial^2 u}{\partial x \partial z})\mathbf{j} + (\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x})\mathbf{k} = \mathbf{0}$ because the mixed derivatives in the brackets cancel out.

(3) $\text{div}\,\text{curl}\,\mathbf{u} = \frac{\partial}{\partial x}(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}) = [(u_3)_{yx} - (u_3)_{xy}] + [-(u_2)_{zx} + (u_2)_{xz}] + [(u_1)_{zy} - (u_3)_{yz}] = 0$.

(4) $\text{div}\,(\text{grad}\, u) = \text{div}\,(\langle u_x, u_y, u_z\rangle) = (u_x)_x + (u_y)_y + (u_z)_z = \Delta u$.

Problem 3. The height of the car at time t is $H(t) := h(x(t), y(t))$. The rate of change of the height is $H'(t) = h_x(\mathbf{x})x'(t) + h_y(\mathbf{x})y'(t) = (\text{grad}\, h)(\mathbf{x}) \cdot \mathbf{x}'(t)$.

Problem 4. $\mathbf{x}(t) = (R\cos(\omega t + \omega_0), R\sin(\omega t + \omega_0))$, so

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t) = R\omega[-\sin(\omega t+\omega_0)f_x(\mathbf{x}) + \cos(\omega t+\omega_0)f_y(\mathbf{x})].$$

Problem 5. Let $E(t) := K(t) + U(t)$. We show that $E(t)$ is constant by showing that its derivative is equal to zero.

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}m\frac{d}{dt}(\mathbf{x}'\cdot\mathbf{x}') + \frac{d}{dt}[U(\mathbf{x})] \\ &= m(\mathbf{x}''\cdot\mathbf{x}') + U_x x' + U_y y' + U_z z' \\ &= m\mathbf{x}''\cdot\mathbf{x}' + \nabla U\cdot\mathbf{x}' = [m\mathbf{x}'' + \nabla U]\cdot\mathbf{x}' \\ &= [-\nabla U + \nabla U]\cdot\mathbf{x}' = \mathbf{0}\cdot\mathbf{x}' = 0 \quad (\because \text{equation of motion}) \end{aligned}$$